Third-party intervention in sequential Wars of Attrition with incomplete information

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Abstract

Vast literature in political science and political economy either argues or assumes that the state's primary role is to provide security. I propose a dynamic model to study the strategic interaction of a state and violent non-state groups that violently compete over rents and when there is uncertainty about the strength of each group. In the model, non-state groups fight sequential Wars of Attrition with replacement under asymmetric information. The violent competition between groups generates a negative externality for the state; the state wants to minimize competition and can temporarily weaken groups' military capabilities by targeting them. The analysis shows that public information about the strength of the last winner of a conflict mediates a trade-off between the likelihood and the intensity of violence when a new group emerges as a challenger; that even if targeting groups is costless for the state, targeting both groups is not always the optimal violence-reducing policy, when the state is weak its optimal policy is to take sides in favor of the incumbent group; and that these differences between weak and strong states will shape the dynamics of reputation building of violent non-state groups.

Keywords: War of Attrition, Third-Party intervention, Reputation, Security

Word count: 5,877

Motivation

Since its foundation, social contract theory has argued that security is either *the* main (Hobbes 1651) or one of the primary functions (Locke 1690) of a state. While this provides a clear and simple principle of how an *ideal type* (Weber 1925; Watkins 1952) of state ought to behave as its citizens' security is jeopardized when non-state groups have a violent competition over valuable resources, it is not straightforward to assess how this principle translates to public security policies. Should this *ideal type* state target its coercive capacity towards all parties involved in the conflict? Will this be the violence-minimizing policy? Upon empirically observing that a state deferentially targets its coercive capacity towards some violent non-state groups and not others, should we immediately conclude that this is behavior that departs from what an *ideal type* of state would do?

The model

The model consists of two periods $t = 1,2$. At each period, t, there are three active players: the state S, and two myopic non-state groups that we will label as the incumbent I and the challenger C. In each period both groups (potentially) fight in a continuous-time *war of attrition*: they compete for a prize by deciding how long they will fight for it. More specifically, within each period t, there is a continuous of dates indexed by $\{t\} \times [0,\infty)$ and each group chooses their quitting point from this set $(t,s) \in \{1,2\} \times [0,\infty)$ where if a group chooses (t,s) it is interpreted as that group having a quitting point s in the conflict at period t. The longer the fight lasts, the higher each group's cost. The side that quits first loses and does not enjoy the prize. The winner of the conflict in the first period $t = 1$, it is labeled as the incumbent group for the second period $t = 2$.

Assume that the conflict between groups have negative externalities for the state and the state prefers conflicts of shorter length. In each period, t, the state can chose to target each individual group by increasing their marginal cost of staying in conflict. The period-t payoffs for an active group j and the state S respectively are given by the following

$$u_j^t := \nu_j \cdot 1_{\{s_j^t > s_{-j}^t\}} - \frac{\zeta_j}{(1-\kappa)^{r_j^t}} \cdot \min\{s_j^t, s_{-j}^t\}$$

$$u_S^t := -\min\{s_j^t, s_{-j}^t\} - K(r_I^t + r_C^t) \cdot \min\{s_j^t, s_{-j}^t\} - 1_{\{r_I^t + r_C^t > 0\}} \cdot \gamma$$

Where $s_j^t \geq 0$ corresponds to the quitting point that group j chooses at period t and $r_I^t \in \{0,1\}$ to the decision of the state on whether to target group j at period t. $\nu_j > 0$ is the value of the price for group j, $\zeta_j > 0$ is the marginal cost of conflict for group j, $\kappa \in (0,1)$ is a measure of state capacity, $\gamma \geq 0$ captures a fixed cost for the state of targeting a group and the cost function K is given by the following.

$$K(n) := \begin{cases} 0 & , n = 0 \\ \beta & , n = 1 \\ \lambda \cdot \beta & , n = 2 \end{cases}$$

Where $\beta \geq 0$ captures the marginal cost of targeting a group as a function of the non-state group's conflict length, the cost function K mediates the costs of targeting different numbers of groups with $K(0) = 0$, $K(1) = \beta$ and $K(2) := \lambda \cdot \beta$, where $\lambda > 1$.

The loser of the conflict in the first period gets a payoff of zero in the second period and the challenger of the second period gets a payoff of zero in the first period.

Assume (ν_j, ζ_j) is private information for group j. The distributional assumptions are given below.

Assumption 1. *The incumbent's ratio of the value of the prize relative the the marginal cost of conflict when the state does not target C, distributes as* $\frac{\nu_C}{\zeta_C} \sim G(\cdot)$*, where G is a cumulative distribution function with the following properties:*

- $Supp[G] = [\theta_{\min}, \theta_{\max}) \subseteq [0, \infty)$*, with* $\theta_{\min} \geq 0$ *and* $\theta_{\max} \in (0, \infty) \cup \{\infty\}$
- $G(0) = 0$
- *G increasing*
- $G'(\theta)$ *exists and it is differentiable for all* $\theta \in [\theta_{\min}, \theta_{\max})$

We will refer to G as the primitive distribution of *types*.

Definition 1. *Define* $G(\cdot|\theta)$ *as the cdf of a random variable* $\theta_I \sim G(\cdot)$ *conditioned on the event* $\{\theta_I \geq \theta_l\}$ *using Bayes rule.*

$$G(\cdot|\theta) := G(\cdot|\theta_l)$$

Assumption 2. *At the beginning of the game,* $\frac{\nu_I}{\zeta_I} \sim G(\cdot|\theta_l)$*, where* $\theta_l \in [\theta_{\min}, \theta_{\max})$.

Assumption 3. $\frac{\nu_I}{\zeta_I}$ *and* $\frac{\nu_C}{\zeta_C}$ *are independent, and independent with all of the other parameters of the game.*

Definition 2. *Define* $\theta_{\min} \mapsto G(\cdot\,; \theta_{\min})$*, where* $G(\cdot\,; \theta_{\min})$ *is the set of all cdfs that satisfy assumption 1.*

In the interest of tractability, we will introduce a new assumption on the primitive distribution of *types* G.

Assumption 4. *The primitive distribution of types* G *satisfies the scale invariance under truncation property:*

$$\theta \sim G(\cdot|\theta_l) \Rightarrow (1-\kappa)\cdot\theta \in G(\cdot\,; (1-\kappa)\cdot\theta_l),\ \forall \theta_l \geq \theta_{\min}$$

Assumption 5. *Whenever* $\theta_j := \frac{v_j}{\zeta_j}$*, with* $j \in \{I, C\}$*, then*

$$\mathbb{E}\left[\min\left\{\int_{\theta_l}^{\theta_I} \theta \cdot \frac{G'(\theta|\theta_l)}{1-G(\theta|\theta_l)} d\theta, \int_{\theta_l}^{\theta_C} \theta \cdot \frac{G'(\theta|\theta_l)}{1-G(\theta|\theta_l)} d\theta\right\} \middle| \theta_C \geq \theta_l\right]$$

exists for all $\theta_l \geq \theta_{\min}$.

The intertemporal payoff for the state is the simple sum of the per-period payoffs

$$U_S := u_S^{t=1} + u_S^{t=2}$$

Both non-state groups are myopic and at period t they only care about the period-t payoff, which implies that at period $t=1$ they ignore the continuation payoffs that they could be receiving at period $t=2$ if they win the conflict at $t=1$.

The timing of events in the game are detailed below.

i) ($t=1$) Nature draws the *types* for the groups

ii) The state chooses $(r_I^{t=1}, r_C^{t=1}) \in \{0,1\}^2$

iii) Groups have a war of attrition

iv) The winner of the war becomes the incumbent

v) ($t=2$) Nature draws the new challenger

vi) The state chooses $(r_I^{t=2}, r_C^{t=2}) \in \{0,1\}^2$

vii) Groups have a war of attrition

The solution concept is Perfect Bayesian equilibrium where there is a positive probability of conflict at both periods. The solution concept rules out the possibility of all types of a non-state group conceding immediately as an equilibrium behavior.

Comments on the model and its assumptions

The lowest quitting point as the measure of violence

In the model, I interpret the lowest quitting point as a measure of the intensity of the violence. I offer two different ways in which this interpretation would make sense. The first and more literal interpretation is that the lowest quitting point corresponds to the time length at which two groups engage in conflict. A more extended conflict period would imply more time at which violence is being used.

The second interpretation is that the evolution of the conflict is indexing the escalation of violence in conflict. Because war is costly, conflicts can escalate to using the most sadistic and cruel ways of violence between non-state groups. Under this interpretation, the lowest quitting point corresponds to the brutality at which the weakest non-state group was willing to tolerate conflict.

Absence of bargaining

In this setting, groups cannot bargain to peacefully split the rents under dispute. This assumption allows for better identification of the mechanisms through which violent non-state groups signal strength through conflict and the effects of state policies that affect their reputation. It is true that violent non-state groups often reach peace agreements among themselves, but because of the lack of a third-party enforcer, a reputation for violence from all parts of conflict is a necessary condition for the agreements to be self-enforcing. So, reputation building is a phenomenon that precludes any possibility of bargaining.

The state's preferences

In the model, the state only cares about minimizing violence and its intervention costs in conflicts with non-state groups. While recognizing that states have a broad range of concerns, including non-state governance and corruption, this paper deliberately narrows the focus to an ideal-type state concerned solely with reducing violence. This approach allows us to explore the optimal security policies for such a state, providing a theoretical basis for comparing how states with different priorities might act. This model serves as a tool to deepen our understanding of state security policy choices in the face of conflict.

Myopic non-state groups and the scale invariance under truncation property

I assume non-state groups are myopic primarily for tractability reasons. This assumption is critical because, without it, no distribution of types makes the equilibrium ratio of the expected value of winning the conflict to the marginal cost of conflict satisfy the scale invariance under truncation property across both periods. The absence of this property leads to significant analytical challenges. Firstly, it disrupts the ability to apply the revenue equivalence theorem (Myerson 1981) to deduce characterizations of the intensive margin by breaking the connection between the equilibrium assignments in a war of attrition and those in a second-price auction. This disruption complicates the derivation of closed-form expressions for the intensive margin under various interventions.

Secondly, and perhaps more problematic, is that the lack of scale invariance under truncation may cause third-party interventions to introduce asymmetries in the distribution of types. Such asymmetries are problematic because they can lead to pathological scenarios where the weaker side in the war of attrition may paradoxically benefit due to the potentially unstable nature of equilibria, as highlighted by other scholars (Krishna 2009; Myerson 2023; Gieczewski 2020).

Equilibrium concept

Perfect Bayesian equilibrium with positive probability of conflict is the appropriate solution concept that would allow us to illustrate and emphasize the role of violence in reputation building and how the reputation of a group can deter conflict. In particular, one of the reasons why Perfect Bayesian equilibrium is the most appropriate equilibrium concept is because groups can update their beliefs about others after they have taken specific actions. Weaker equilibrium concepts such as *Bayesian Nash equilibrium* or *subgame perfect equilibrium* would either fail to incorporate the role that reputation has in deterring violence between groups (this because *sequential rationality* would be violated) or would generate reputation in equilibrium that would violate Bayes rule (this because *weakly consistency* would be violated).

Moreover the focus on equilibria where there is a positive probability of conflict at both periods is justified by the motivation of this paper. If we want to understand how a violence-minimizing chooses among different targeting policies we should focus in situations where conflict might occur. If there is no possibility of conflict then all policies are equally effective.

Further refinements of Perfect Bayesian equilibrium are not necessary because the myopic preferences of non-state groups coupled with the assumptions on the primitive distribution of types will imply that there is a unique equilibrium and where all information sets are included in the support of outcomes that occur *on-the-path of play*.

Definitions and terminology

Denote θ_l as "the reputation of the incumbent" at $t = 1$, and if it is public information at $t = 2$ that $\theta_I \geq \theta_l'$ we will say that θ_l' is the reputation of the incumbent at $t = 2$.

Let $(\sigma_I^t[h], \sigma_C^t[h])$ be a strategy profile for groups at period t

Denote "violence at t", v^t, as

$$v^t[h] := \min\left\{\sigma_I^t[h](\theta_I), \sigma_C^t[h](\theta_C)\right\}$$

Define the "extensive margin at t", EM^t, as

$$EM^t[h] := Pr[v^t[h] > 0|h]$$

Define the "intensive margin at t", IM^t, as

$$IM^t[h] := \mathbb{E}[v^t|v^t[h] > 0]$$

Denote as the "expected violence at t", EV^t, as

$$EV^t[h] := \mathbb{E}[v^t[h]|h] = EM^t[h] \cdot IM^t[h]$$

Analysis

The analysis unfolds in the following way. First, we establish that the Pareto distribution is the singular family of distributions consistent with our assumptions regarding the primitive distribution of types. Following this, we proceed to characterize the unique equilibrium in the subgame in the second period, where the state opts against targeting any group. This equilibrium serves as a critical benchmark, enabling us to comprehensively derive the equilibrium conflict behavior of non-state groups across both periods under any permutation of enforcement policies. Subsequently, we focus on characterizing the optimal policy in the second period, considering scenarios both with and without the inclusion of enforcement policy costs. This analysis provides nuanced insights into the strategic considerations influencing state decisions. Finally, we identify the optimal equilibrium targeting policy that the state chooses in the first period.

Lemma 1. *Assumptions 1 and 4 are satisfied if and only if the primitive distribution of types G corre-*

spond to a Pareto distribution with scale parameter $\theta_{\min}$ and shape parameter $\alpha > 0$.

$$G(\theta) = \begin{cases} 1 - \left(\frac{\theta_{\min}}{\theta}\right)^{\alpha} & , \theta_{\min} \leq \theta \\ 0 & , \theta_{\min} > \theta \end{cases}$$

$$G(\theta;\theta_l) = G(\theta|\theta_I \geq \theta_l) = \begin{cases} 1 - \left(\frac{\theta_l}{\theta}\right)^{\alpha} & , \theta_l \leq \theta \\ 0 & , \theta_l > \theta \end{cases}$$

Furthermore, when assumptions 2, 3 and 5 are included this implies that $\alpha > \frac{1}{2}$.

Notation 1. *Fix $\theta_{\min} > 0$ and $\alpha \in \left(\frac{1}{2}, \infty\right)$, then we will refer to the cdf of the Pareto distribution with scale parameter $\theta_{\min} > 0$ and shape parameter $\alpha \in \left(\frac{1}{2}, \infty\right)$ as G. If $(\theta_l, \alpha') \neq (\theta_{\min}, \alpha)$ then we will refer to the cdf of the Pareto distribution with scale parameter $\theta_l > 0$ and shape parameter $\alpha' \in \left(\frac{1}{2}, \infty\right)$ as $G_{\theta_l,\alpha'}$.*

Equilibrium conflict behavior without intervention at $t = 2$

The next proposition characterizes the equilibrium conflict behavior in the second period when the state does not target any group.

Proposition 1. *In the subgame at $t = 2$, after the state chooses not to intervene $(r_I, r_C) = (0,0)$ and it is common knowledge that $\theta_I > \theta_l'$, then there is a unique equilibrium where both groups choose their stopping point according to strategy $\sigma[\theta_l']$*

$$\sigma[\theta_l'](\theta) = \begin{cases} \alpha \cdot \left(\theta - \theta_l'\right) & , \theta > \theta_l' \\ 0 & , \theta \leq \theta_l' \end{cases}$$

The extensive and intensive margin and the expected violence are given by the following

$$EM_{00}^{t=2}[\theta_l'] = \left(\frac{\theta_{\min}}{\theta_l}\right)^{\alpha}$$
$$IM_{00}^{t=2}[\theta_l'] = \frac{\alpha}{2 \cdot \alpha - 1} \cdot \theta_l$$
$$EV_{00}^{t=2}[\theta_l'] = \frac{\alpha}{2 \cdot \alpha - 1} \cdot \theta_{\min}^{\alpha} \cdot \theta_l^{1-\alpha}$$

As θ_l' increases, $EM_{00}^{t=2}[\theta_l']$ decreases, $IM_{00}^{t=2}[\theta_l']$ increases and $EV_{00}^{t=2}[\theta_l']$ is increasing when $\alpha \in$

$\left(\frac{1}{2}, 1\right)$*, constant when* $\alpha = 1$ *and decreasing when* $\alpha \in (1, \infty)$.

In the context of the war of attrition models, a common challenge encountered is the multiplicity of equilibria. However, when we introduce the assumption that the primitive distribution of types follows a Pareto distribution, we find that this leads to a unique equilibrium. In this unique equilibrium, both the incumbent and the challengers are modeled to adopt identical strategies in terms of their engagement in conflict. However, despite this symmetry in strategies, a fundamental asymmetry underlies their actions due to the distribution of types. Notably, the lower bound of the support for the challengers' types is weakly lower than that of the incumbent's types. This distinction is important because it implies that, in equilibrium, there exist certain types of challengers who opt not to engage in conflict, while incumbents fight with probability one.

In equilibrium, the expected violence as a function of the reputation of the incumbent group depends on distributional assumptions, particularly the shape of the distribution. This interplay is characterized by two opposing forces. On the one hand, a higher reputation for violence on the part of the incumbent non-state actor serves as a deterrent to weak challengers, effectively reducing the expected violence in equilibrium. On the other hand, the model also suggests an intensification effect; that is, should a conflict arise, the presence of an incumbent with a larger reputation for violence predicts more intense conflicts. This intensification effect operates on the intensive margin, indicating that conditional upon the outbreak of conflict, the severity of the conflict escalates with the incumbent's reputation.

The predominance of either the deterrent effect or the intensification effect hinges critically on the shape parameter of the Pareto distribution. Specifically, for lower values of the shape parameter, the Pareto distribution exhibits a heavier tail, and the intensive margin—pertaining to the intensity of conflict—dominates as the incumbent's reputation increases. Conversely, for higher values of the shape parameter, the distribution becomes less heavy-tailed, indicating that the extensive margin—reflecting the deterrent effect on conflict—overshadows the intensive margin. In this scenario, an increase in the incumbent's reputation for violence leads to a decrease in the expected level of violence, as the potential for deterrence outweighs the propensity for more severe conflicts. This nuanced relationship between the shape parameter of the Pareto distribution and the equilibrium outcomes of violence underscores the parameter's critical role in determining the state's preference towards non-state groups with either high or low reputations for violence. It hints at a strategic calculus that the state must navigate when choosing whom to target policies that might change the distribution of reputation in the future.

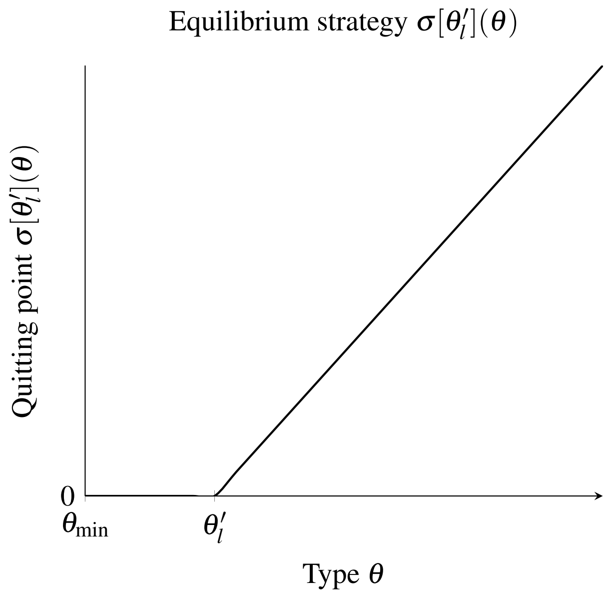

Figure 1: The graph shows equilibrium strategy after the state does not target any group, $\sigma[\theta_l'](\theta)$, as a function of the type θ. The graph was generated using $\theta_{\min} = 1$, $\theta_l' = 2$ and $\alpha = 1$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$, $\theta_l' \in [\theta_{\min}, \infty)$ and $\alpha \in \left(\frac{1}{2}, \infty\right)$.

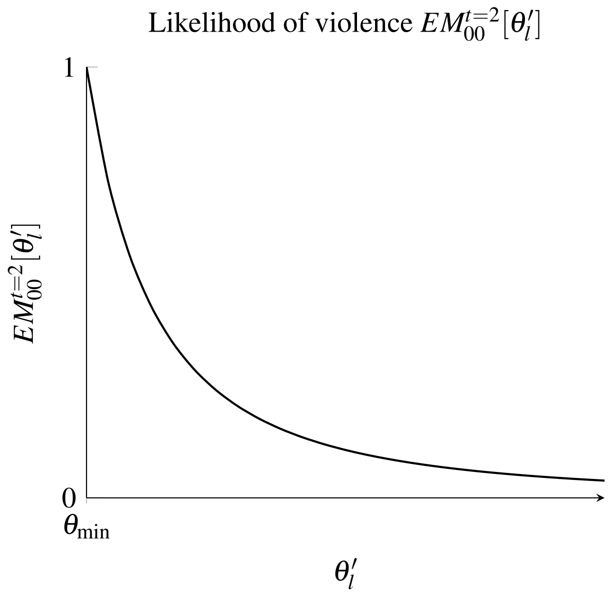

Figure 2: The graph shows the likelihood of violence at $t = 2$ after the state chose to not target any group $EM_{00}^{t=2}[\theta_l'] = \left(\frac{\theta_{\min}}{\theta_l}\right)^{\alpha}$ as a function of the reputation of the incumbent θ_l'. The graph was generated using $\theta_{\min} = 1$, $\alpha = 2$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$ and $\alpha \in \left(\frac{1}{2}, \infty\right)$.

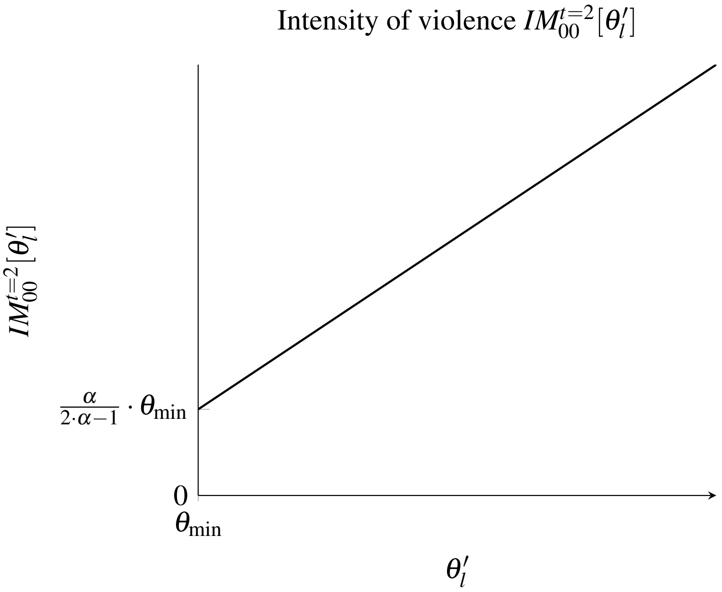

Figure 3: The graph shows the intensity of violence at $t=2$ after the state chose to not target any group $IM_{00}^{t=2}[\theta_l'] = \left(\frac{\theta_{\min}}{\theta_l}\right)^{\alpha}$ as a function of the reputation of the incumbent θ_l'. The graph was generated using $\theta_{\min}=1$, $\alpha=1$, but the qualitative properties of the graph do not change for different values of $\theta_{\min}>0$ and $\alpha \in \left(\frac{1}{2},\infty\right)$.

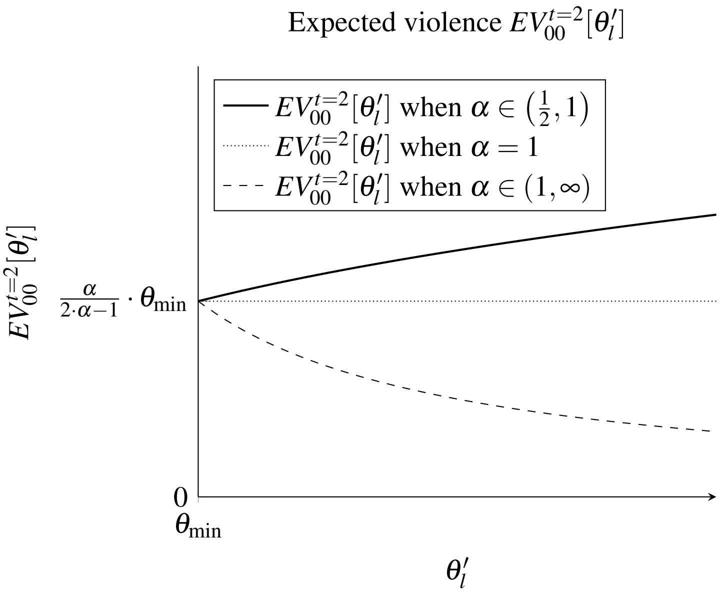

Figure 4: The graph shows the expected violence at $t=2$ after the state chose to not target any group $EV_{00}^{t=2}[\theta_l'] = \frac{\alpha\cdot\theta_{\min}^{\alpha}}{2\cdot\alpha-1}\cdot\theta_l^{1-\alpha}$ as a function of the reputation of the incumbent θ_l'. The graphs were generated using $\theta_{\min}=1$, $\alpha=\frac{2}{3}$ for the solid graph and $\alpha=2$ for the dashed graph, but the qualitative properties of the graph do not change for different values of $\theta_{\min}>0$, $\alpha\in\left(\frac{1}{2},1\right)$ and $\alpha\in(1,\infty)$ respectively.

Equilibrium conflict behavior with intervention at $t = 2$

The next proposition characterizes the equilibrium conflict behavior under different state targeting policies.

Proposition 2. *In the subgame where at* $t = 2$*, the incumbent group* I *has reputation* $\theta_l' \geq \theta_{\min}$ *and after the state chose targeting profile* $\left(r_I^{t=2}, r_C^{t=2}\right) = (r_I, r_C)$*, then there is a unique equilibrium where both groups choose their stopping points according to strategies given by the following*

$$\sigma_I^{t=2}[r_I, r_C, \theta_l](\theta_I) = \begin{cases} \sigma[\theta_l'](\theta_I) & , (r_I, r_C) \in \{(0,0),(0,1)\} \\ \sigma[\max\{\theta_l' \cdot (1-\kappa), \theta_{\min}\}](\theta_I \cdot (1-\kappa)) & , (r_I, r_C) = (1,0) \\ \sigma[\theta_l' \cdot (1-\kappa)](\theta_I \cdot (1-\kappa)) & , (r_I, r_C) = (1,1) \end{cases}$$

$$\sigma_C^{t=2}[r_I, r_C, \theta_l'](\theta_C) = \begin{cases} \sigma[\theta_l'](\theta_C) & , (r_I, r_C) = (0,0) \\ \sigma[\theta_l'](\theta_C \cdot (1-\kappa)) & , (r_I, r_C) = (0,1) \\ \sigma[\max\{\theta_l' \cdot (1-\kappa), \theta_{\min}\}](\theta_C) & , (r_I, r_C) = (1,0) \\ \sigma[\theta_l' \cdot (1-\kappa)](\theta_C \cdot (1-\kappa)) & , (r_I, r_C) = (1,1) \end{cases}$$

The extensive margin is given by the following

$$EM_{r_I r_C}^{t=2}[\theta_l'] = \begin{cases} \left(\frac{\theta_{\min}}{\theta_l'}\right)^{\alpha} & , (r_I, r_C) \in \{(0,0),(1,1)\} \\ (1-\kappa)^{\alpha} \cdot \left(\frac{\theta_{\min}}{\theta_l'}\right)^{\alpha} & , (r_I, r_C) = (0,1) \\ \min\left\{\left(\frac{\theta_{\min}}{(1-\kappa)\cdot\theta_l'}\right)^{\alpha}, \left(\frac{(1-\kappa)\cdot\theta_l'}{\theta_{\min}}\right)^{\alpha}\right\} & , (r_I, r_C) = (1,0) \end{cases}$$

The intensive margin is given by the following

$$IM_{r_I r_C}^{t=2}[\theta_l'] = \begin{cases} \frac{\alpha}{2\cdot\alpha-1} \cdot \theta_l' & , (r_I, r_C) \in \{(0,0),(0,1)\} \\ \frac{\alpha}{2\cdot\alpha-1} \cdot (1-\kappa) \cdot \theta_l' & , (r_I, r_C) = (1,1) \\ \frac{\alpha}{2\cdot\alpha-1} \cdot \max\left\{(1-\kappa) \cdot \theta_l', \theta_{\min}\right\} & , (r_I, r_C) = (1,0) \end{cases}$$

The expected violence is given by the following

$$EV^{t=2}_{r_I r_C}[\theta'_l] = \begin{cases} \frac{\alpha}{2\alpha-1} \cdot \theta^{\alpha}_{\min} {\theta'_l}^{1-\alpha} & , (r_I, r_C) = (0,0) \\ \frac{\alpha}{2\alpha-1} \cdot (1-\kappa)^{\alpha} \cdot \theta^{\alpha}_{\min} {\theta'_l}^{1-\alpha} & , (r_I, r_C) = (0,1) \\ \frac{\alpha}{2\alpha-1} \cdot \min\left\{ \left(\frac{\theta_{\min}}{(1-\kappa)\cdot\theta'_l}\right)^{\alpha}, \left(\frac{(1-\kappa)\cdot\theta'_l}{\theta_{\min}}\right)^{\alpha} \right\} \cdot \max\left\{(1-\kappa)\cdot\theta'_l, \theta_{\min}\right\} & , (r_I, r_C) = (1,0) \\ \frac{\alpha}{2\alpha-1} \cdot (1-\kappa) \cdot \theta^{\alpha}_{\min} {\theta'_l}^{1-\alpha} & , (r_I, r_C) = (1,1) \end{cases}$$

Moreover, targeting both groups $(r_I, r_C) = (1,1)$ *is the policy that minimizes the expected violence if and only if* $\alpha < 1$*; and targeting only the challenger* $(r_I, r_C) = (0,1)$ *is the policy that minimizes the expected violence if and only if* $\alpha > 1$.

Leveraging the property of scale invariance under truncation, our model introduces a novel approach to model conflicts involving third-party interventions, particularly in settings characterized by asymmetric information. This property allows for the construction of isomorphisms between subgames where the state actively targets one or more groups to subgames where no group is targeted, and the distribution of types is scaled down. This transformation preserves the strategic essence of the conflict while adjusting the parameters to reflect the impact of state intervention on the actors' type distributions and its effects in a tractable way.

In the model, when the incumbent's reputation is sufficiently weak relative to the state's capacity and the scale parameter of the primitive distribution of types, state intervention in the form of targeting only the incumbent diminishes the incumbent's relative strength to the point where there is a strategic reversal in the equilibrium behavior of the involved parties. Essentially, the incumbent becomes so weakened by the targeted state intervention that its position and strategic behavior in the conflict mirror what would typically be expected of a challenger. In this case, there will be types of incumbents who will give up the prize without a fight, and the challengers will fight with probability one.

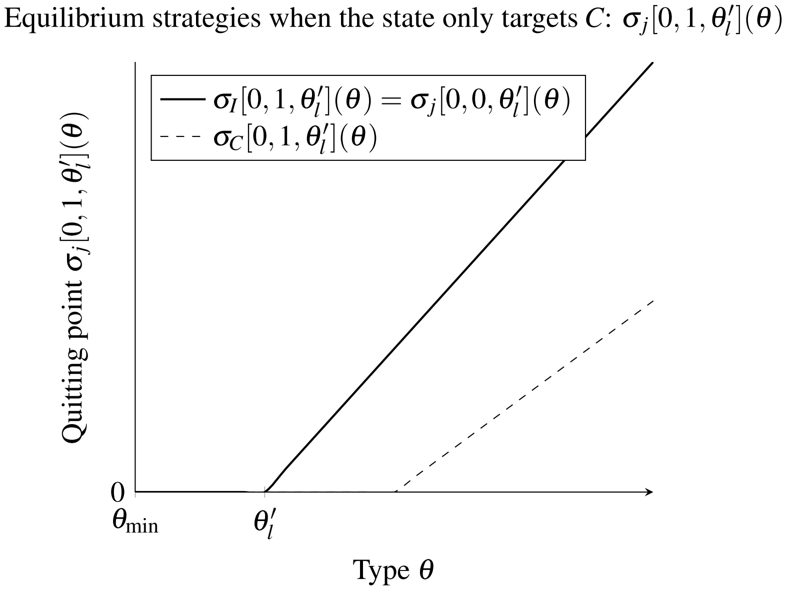

Figure 5: The graph shows equilibrium strategies after the state only targets the challenger, $\sigma_j[0,1,\theta_l'](\theta)$, as functions of the type θ. The graph was generated using $\theta_{\min} = 1$, $\theta_l' = 2$, $\alpha = 1$ and $\kappa = \frac{1}{3}$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$, $\theta_l' \in [\theta_{\min}, \infty)$, $\alpha \in \left(\frac{1}{2}, \infty\right)$ and $\kappa \in (0,1)$.

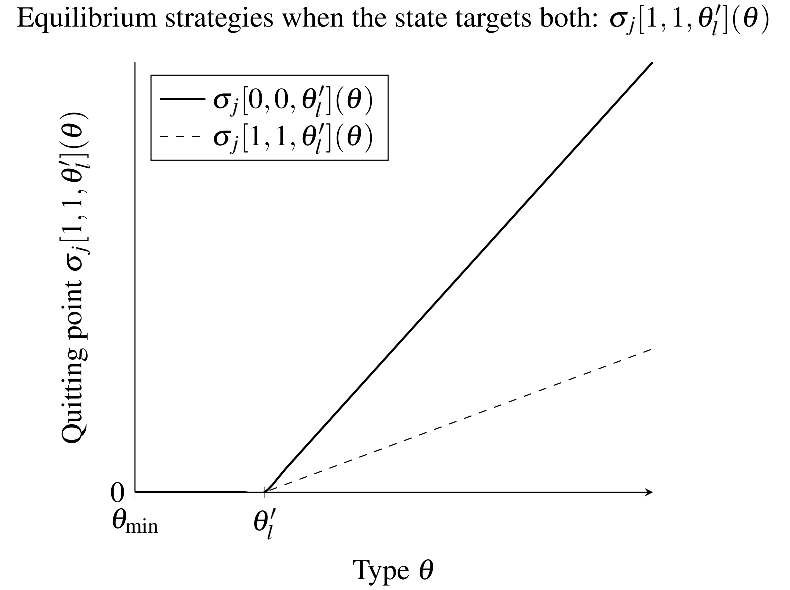

Figure 6: The graph shows equilibrium strategies after the state targets both groups, $\sigma_j[1,1,\theta_l'](\theta)$, as functions of the type θ. The graph was generated using $\theta_{\min} = 1$, $\theta_l' = 2$, $\alpha = 1$ and $\kappa = \frac{1}{3}$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$, $\theta_l' \in [\theta_{\min}, \infty)$, $\alpha \in \left(\frac{1}{2}, \infty\right)$ and $\kappa \in (0,1)$.

Equilibrium strategies when the state targets both: $\sigma_j[1,0,\theta_l'](\theta)$ when $\theta_l' \cdot (1-\kappa) > \theta_{\min}$

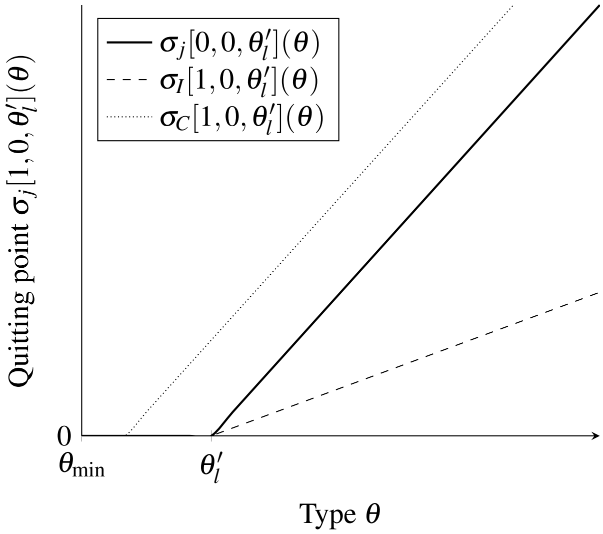

Figure 7: The graph shows equilibrium strategies after the state targets only the incumbent, $\sigma_j[1,0,\theta_l'](\theta)$, as functions of the type θ when the state is not strong enough $\theta_l' \cdot (1-\kappa) > \theta_{\min}$. The graph was generated using $\theta_{\min} = 1$, $\theta_l' = 2$, $\alpha = 1$ and $\kappa = \frac{1}{3}$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$, $\theta_l' \in [\theta_{\min}, \infty)$, $\alpha \in \left(\frac{1}{2}, \infty\right)$, $\kappa \in (0,1)$ and $\theta_l' \cdot (1-\kappa) > \theta_{\min}$.

Equilibrium strategies when the state targets both: $\sigma_j[1,0,\theta_l'](\theta)$ when $\theta_l' \cdot (1-\kappa) < \theta_{\min}$

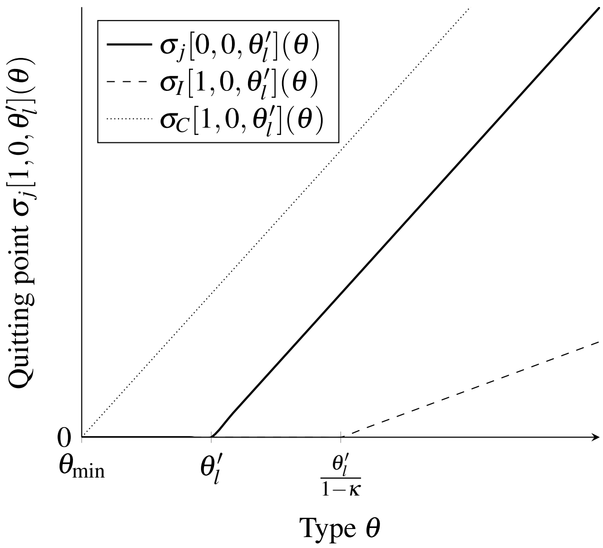

Figure 8: The graph shows equilibrium strategies after the state targets only the incumbent, $\sigma_j[1,0,\theta_l'](\theta)$, as functions of the type θ when the state is not strong enough $\theta_l' \cdot (1-\kappa) < \theta_{\min}$. The graph was generated using $\theta_{\min} = 1$, $\theta_l' = 2$, $\alpha = 1$ and $\kappa = \frac{2}{3}$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$, $\theta_l' \in [\theta_{\min}, \infty)$, $\alpha \in \left(\frac{1}{2}, \infty\right)$, $\kappa \in (0,1)$ and $\theta_l' \cdot (1-\kappa) < \theta_{\min}$.

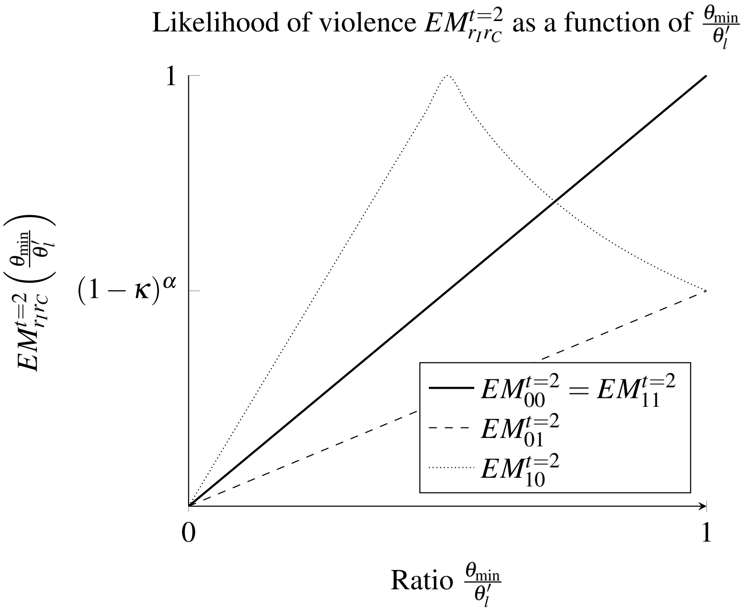

Figure 9: The graph shows the likelihood of violence at $t=2$ for all the policies that the state can choose $EM^{t=2}_{00}$, $EM^{t=2}_{01}$, $EM^{t=2}_{10}$ and $EM^{t=2}_{11}$ as a function of the ratio $\frac{\theta_{\min}}{\theta'_l}$. The graph was generated using $\alpha = 1$ and $\kappa = \frac{1}{2}$, but the qualitative properties of the graph do not change for different values of $\alpha \in \left(\frac{1}{2}, \infty\right)$ and $\kappa \in (0,1)$.

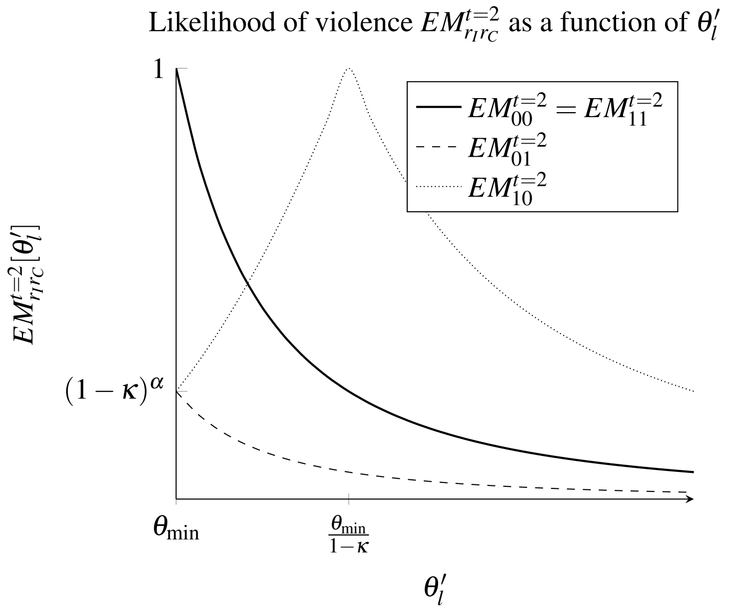

Figure 10: The graph shows the likelihood of violence at $t=2$ for all the policies that the state can choose $EM^{t=2}_{00}$, $EM^{t=2}_{01}$, $EM^{t=2}_{10}$ and $EM^{t=2}_{11}$ as a function of the reputation of the incumbent θ'_l. The graph was generated using $\theta_{\min} = 1$, $\alpha = 2$ and $\kappa = \frac{1}{2}$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$, $\alpha \in \left(\frac{1}{2}, \infty\right)$ and $\kappa \in (0,1)$.

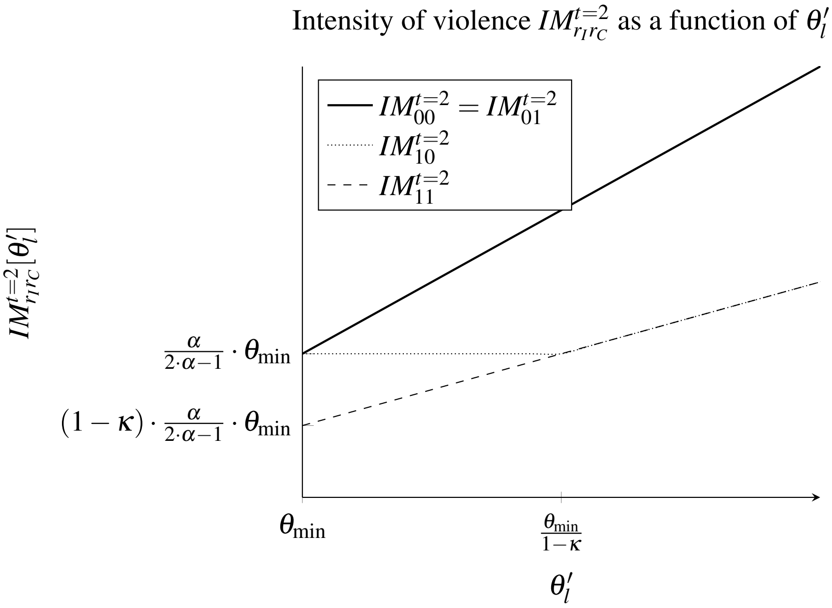

Figure 11: The graph shows the intensity of violence at $t = 2$ for all the policies that the state can choose $IM^{t=2}_{00}$, $IM^{t=2}_{01}$, $IM^{t=2}_{10}$ and $IM^{t=2}_{11}$ as a function of the reputation of the incumbent θ'_l. The graph was generated using $\theta_{\min} = 1$, $\alpha = 1$ and $\kappa = \frac{1}{2}$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$, $\alpha \in \left(\frac{1}{2}, \infty\right)$ and $\kappa \in (0, 1)$.

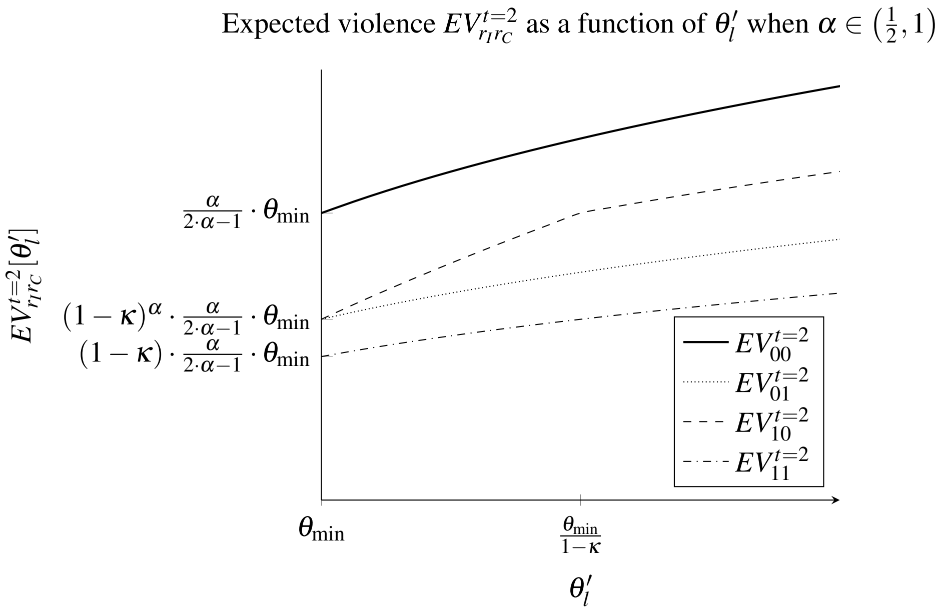

Figure 12: The graph shows the expected violence at $t = 2$ for all the policies that the state can choose $EV^{t=2}_{00}$, $EV^{t=2}_{01}$, $EV^{t=2}_{10}$ and $EV^{t=2}_{11}$ as a function of the reputation of the incumbent θ'_l when $\alpha \in \left(\frac{1}{2}, 1\right)$. The graph was generated using $\theta_{\min} = 1$, $\alpha = \frac{2}{3}$ and $\kappa = \frac{1}{2}$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$, $\alpha \in \left(\frac{1}{2}, 1\right)$ and $\kappa \in (0, 1)$.

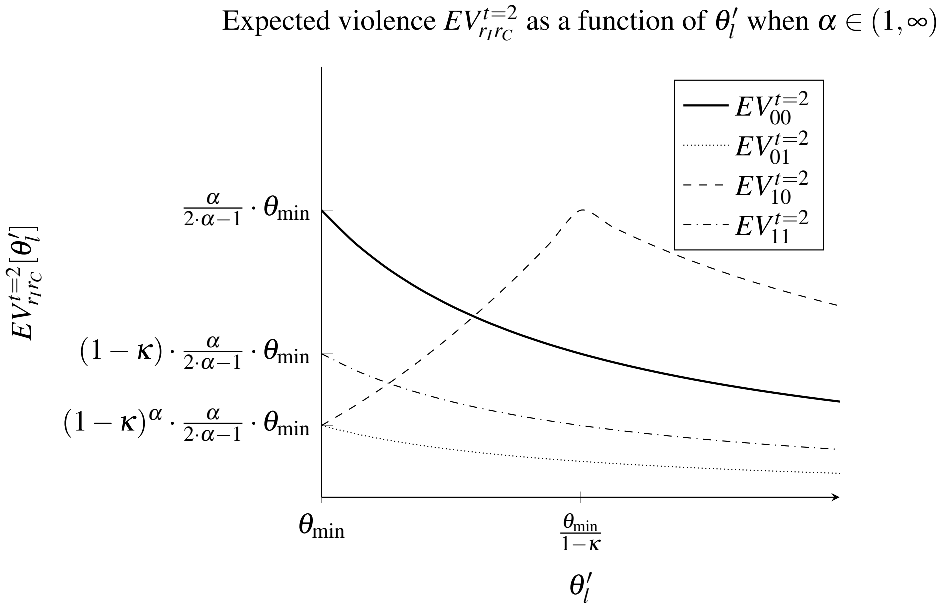

Figure 13: The graph shows the expected violence at $t = 2$ for all the policies that the state can choose $EV_{00}^{t=2}$, $EV_{01}^{t=2}$, $EV_{10}^{t=2}$ and $EV_{11}^{t=2}$ as a function of the reputation of the incumbent θ_I' when $\alpha \in (1,\infty)$. The graph was generated using $\theta_{\min} = 1$, $\alpha = 2$ and $\kappa = \frac{1}{2}$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$, $\alpha \in (1,\infty)$ and $\kappa \in (0,1)$.

Definition 3. *Define the "expected violence production function" $F : (0,\infty)^2 \to (0,\infty)$ as*

$$(\theta_1,\theta_2) \mapsto F[\theta_1,\theta_2] := \frac{\alpha}{2\cdot\alpha-1}\cdot(\min\{\theta_1,\theta_2\})^{\alpha}\cdot(\max\{\theta_1,\theta_2\})^{1-\alpha}$$

Introducing the concept of an "expected violence production function" within our analysis will be very useful, as it allows for a consistent and intuitive expression of expected violence in any given period as a function of two key variables: the incumbent's reputation and the scale parameter of the distribution of types. Importantly, this relationship is modulated by a constant factor, providing a clear framework for understanding the dynamics of violence. Notably, when the shape parameter of the distribution is less than one, this expected violence production function aligns precisely with the Cobb-Douglas production function familiar in economics. In this context, the scale parameter and the incumbent's reputation act as complements, enhancing the production of violence in equilibrium. Conversely, a shape parameter greater than one signifies a scenario where an increased reputation of the incumbent predominantly serves a deterrent role, effectively lowering the expected levels of violence.

Corollary 1. *The expected violence at t when the targeting policy is $\left(r_I^t, r_C^t\right) = (r_I, r_C)$ is given by the following expression.*

$$EV_{r_I,r_C}[\theta_l] = F\left[(1-\kappa)^{r_I}\theta_l, (1-\kappa)^{r_C}\theta_{\min}\right] = \begin{cases} F[\theta_l, \theta_{\min}] & , (r_I, r_C) = (0,0) \\ F[\theta_l, (1-\kappa)\cdot\theta_{\min}] & , (r_I, r_C) = (0,1) \\ F[(1-\kappa)\cdot\theta_l, \theta_{\min}] & , (r_I, r_C) = (1,0) \\ (1-\kappa)\cdot F[\theta_l, \theta_{\min}] & , (r_I, r_C) = (1,1) \end{cases}$$

Optimal targeting policy at $t = 2$

The next corollary provides the foundation to analyze the cost-benefit analysis that the state has when choosing its optimal policy.

Corollary 2. *If targeting groups is costless for the state, $\beta = \gamma = \lambda = 0$, then the optimal targeting policy for the state at $t = 2$ is given by the following.*

$$\left(r_I^{*t=2}\left[\theta_l'\right], r_C^{*t=2}\left[\theta_l'\right]\right) = \begin{cases} (1,1) & , \alpha \in \left(\frac{1}{2}, 1\right) \\ \{(0,1),(1,1)\} & , \alpha = 1 \\ (0,1) & , \alpha \in (1,\infty) \end{cases}$$

In the scenario where targeting groups incur no cost to the state, the decision on whether state intervention reduces violence hinges on two primary considerations. Firstly, the act of targeting by the state influences groups to adopt a lower quitting point in equilibrium, meaning that groups are more likely to desist from conflict earlier than they would in the absence of targeting. This dynamic suggests a potential strategy for the state to target both groups to uniformly lower their propensity for prolonged engagement, thus reducing overall violence. However, differentially targeting groups can amplify the deterrent effect on the targeted group, particularly discouraging weaker types within that group from initiating conflict. This selective targeting strategy might, under certain conditions, be more effective in reducing violence by increasing the cost of entry into conflict for these weaker types.

The determinative factor in choosing between uniformly targeting both groups or selectively targeting one lies in the shape parameter of the primitive distribution of types. When the distribution exhibits a heavier tail—indicating a larger presence of extreme types—the state finds it advantageous to target both groups, as this broad action dampens the potential for violence. Conversely, in distributions with lighter tails, where extreme types are less prevalent, only targeting the challenger emerges as the preferred strategy. This approach effectively raises the barrier to conflict entry, leveraging the deterrent effect to avert potential engagements by making the cost of initiating conflict prohibitive for all but the strongest challengers. Thus, the shape parameter crucially mediates the state's strategic decision, influencing whether a more uniform or differential targeting strategy is adopted to minimize violence.

Lemma 2. *For any $\theta_l' > \theta_{\min}$, targeting only the challenger strictly dominates targeting only the incumbent at the second period $t = 2$.*

In the absence of future considerations in the second period, the analysis reveals that targeting only the incumbent results in higher expected violence compared to scenarios where only the challenger is targeted. This outcome stems from the incumbent's critical role in the equilibrium dynamics of conflict; when the incumbent is specifically weakened through targeting, it inadvertently encourages a broader spectrum of challengers to engage, perceiving an increased opportunity for success. Conversely, targeting only the challenger tends to deter potential challengers due to a heightened risk of state intervention, thereby reducing the likelihood of conflict initiation.

Proposition 3. *There exist, threshold functions $\kappa^{**}(\beta,\lambda,\alpha)$ and $F^{**}(\kappa,\beta,\lambda,\gamma,\alpha)$, such that the optimal targeting policy for the state at $t = 2$ is given by the following.*

$$\left(r_I^{*t=2}\left[\theta_l'\right], r_C^{*t=2}\left[\theta_l'\right]\right) = \begin{cases} (0,0) & , F\left[\theta_l, \theta_{\min}\right] < F^{**}(\kappa,\beta,\lambda,\gamma,\alpha) \\ (0,1) & , F\left[\theta_l, \theta_{\min}\right] > F^{**}(\kappa,\beta,\lambda,\gamma,\alpha) \text{ and } \kappa < \kappa^{**}(\beta,\lambda,\alpha) \\ (1,1) & , F\left[\theta_l, \theta_{\min}\right] > F^{**}(\kappa,\beta,\lambda,\gamma,\alpha) \text{ and } \kappa > \kappa^{**}(\beta,\lambda,\alpha) \end{cases}$$

Where κ^{**} and F^{**} are given by the following.

$$\kappa^{**}(\beta,\lambda,\alpha) := \begin{cases} 1 & , \alpha \geq 1 \\ 1 - \sqrt[1-\alpha]{\frac{\beta+1}{\lambda\beta+1}} & , \alpha < 1 \end{cases}$$

$$F^{**}(\kappa,\beta,\lambda,\gamma,\alpha) := \min\left\{F_{01}^{**}(\kappa,\beta,\gamma,\alpha), F_{11}^{**}(\kappa,\beta,\gamma,\lambda)\right\}$$

Where F_{01}^{**} and F_{11}^{**} are defined as follows.

$$F_{01}^{**}(\kappa,\beta,\gamma,\alpha) := \begin{cases} \frac{\gamma}{1-(1-\kappa)^{\alpha}(\beta+1)} & , (1-\kappa)^{\alpha}(\beta+1) < 1 \\ \infty & , (1-\kappa)^{\alpha}(\beta+1) \geq 1 \end{cases}$$

$$F_{11}^{**}(\kappa,\beta,\gamma,\lambda) := \begin{cases} \frac{\gamma}{1-(1-\kappa)(\lambda\beta+1)} & , (1-\kappa)(\lambda\beta+1) < 1 \\ \infty & , (1-\kappa)(\lambda\beta+1) \geq 1 \end{cases}$$

In scenarios where targeting policies entail costs, the determination of the optimal enforcement strategy is intricately linked to the shape parameter of the distribution of types. Specifically, when the shape parameter exceeds 1, the optimal policy hinges on the state's capacity and the expected violence, which in turn depends on the incumbent's reputation and the scale parameter of the distribution of types. Qualitatively, in situations where expected violence is relatively low, and the state's capacity is low, intervention is inefficient. Conversely, in contexts where expected violence and state capacity are high enough, the state's optimal choice shifts towards intervening by exclusively targeting the challenger. This strategy aims to maximize the deterrence effect while managing the costs associated with enforcement actions.

Conversely, with a low shape parameter indicating a heavier tail in the primitive distribution of types, the decision-making framework for state intervention mirrors the initial assessment but introduces state capacity as a critical mediator. The state could find it optimal to target only the challenger or to adopt a uniform targeting approach against both groups. Here, the pivotal factor is state capacity; higher capacity leans the state's preference toward uniform targeting.

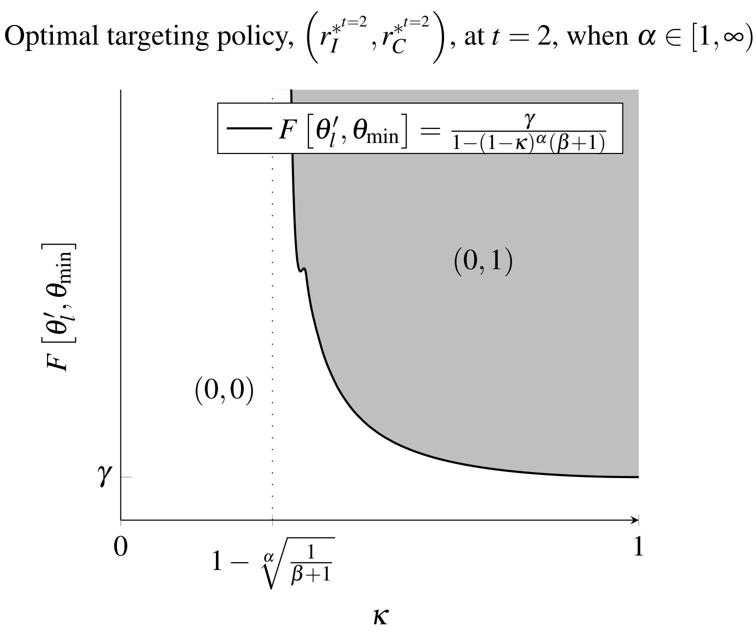

Figure 14: The graph shows the optimal targeting policy, $\left(r_I^{*^{t=2}}[\theta_l'], r_C^{*^{t=2}}[\theta_l']\right)$, at $t=2$, when $\alpha \geq 1$ for different values of $F\left[\theta_l', \theta_{\min}\right]$ and state capacity κ. The graph was generated using $\alpha = 2$, $\beta = 1$ and $\gamma = 1$, but the qualitative properties of the graph do not change for different values of $\alpha \in [1,\infty)$, $\beta > 0$ and $\gamma > 0$.

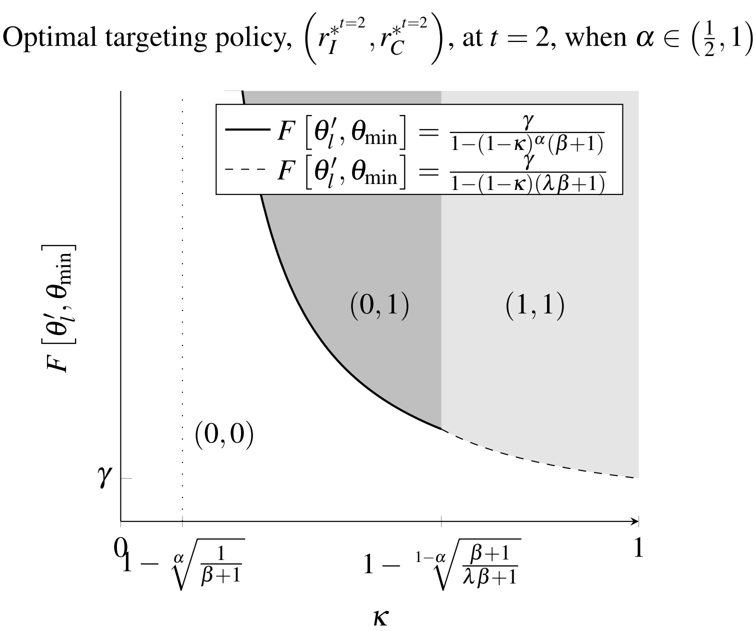

Figure 15: The graph shows the optimal targeting policy, $\left(r_I^{*^{t=2}}[\theta_l'], r_C^{*^{t=2}}[\theta_l']\right)$, at $t=2$, when $\alpha \in \left(\frac{1}{2},1\right)$ for different values of $F\left[\theta_l', \theta_{\min}\right]$ and state capacity κ. The graph was generated using $\alpha = \frac{3}{4}$, $\beta = \frac{1}{10}$, $\gamma = 1$ and $\lambda = 4$, but the qualitative properties of the graph do not change for different values of $\alpha \in [1,\infty)$, $\beta > 0$, $\gamma > 0$ and $\lambda > 1$.

Optimal targeting policy at $t = 1$

The subsequent sections of this paper will delve into the strategic decisions faced by the state in the first period, encompassing efforts to mitigate violence not only in the present but also considering the effects that this can have on the equilibrium behavior in the second period. This analysis will highlight that, under certain parameter configurations, the state encounters a nuanced trade-off between minimizing immediate violence and considering the strategic implications of its policies on future conflict dynamics. Specifically, the information conveyed through the state's actions influences the reputation of the conflict's winner, which, in turn, significantly impacts the landscape of violence in the next period.

Definition 4. *Let $r_1, r_2 \in \{0,1\}$. Denote $\theta_l' | \{(r_I^{t=1}, r_C^{t=1}) = (r_1, r_2)$ as $\theta_l' | r_1 r_2$.*

Lemma 3. *In the subgame where at $t = 1$, the incumbent group I has reputation $\theta_l \geq \theta_{\min}$ and after the state chose $(r_I^{t=1}, r_C^{t=1}) \in \{0,1\}^2$, then the distribution of the reputation of the incumbent at $t = 2$, θ_l', will be given by the following*

$$\theta_l'|00 \sim \theta_l'|11 \sim \min\{\theta_I, \max\{\theta_C, \theta_l\}\}$$

$$\theta_l'|01 \sim \begin{cases} \frac{\theta_I}{1-\kappa} & , \ \theta_C \cdot (1-\kappa) > \theta_I \\ \max\{\theta_l, \theta_C \cdot (1-\kappa)\} & , \ \theta_C \cdot (1-\kappa) < \theta_I \end{cases}$$

$$\theta_l'|10 \sim \begin{cases} \max\{\theta_{\min}, \theta_I \cdot (1-\kappa)\} & , \ \theta_C \cdot (1-\kappa) > \theta_I \\ \max\left\{\theta_l, \frac{\theta_C}{1-\kappa}\right\} & , \ \theta_C \cdot (1-\kappa) < \theta_I \end{cases}$$

Moreover, the conditional cdfs are given by the following

$$G_{\theta_l'|00}(\theta) =$$

$$G_{\theta_l'|11}(\theta) = 1_{\{\theta \geq \theta_l\}} G(\theta_l) + (1 - G(\theta_l)) \cdot G_{\theta_l, 2\cdot\alpha}(\theta)$$

$$G_{\theta_l'|01}(\theta) = 1_{\{\theta \geq \theta_l\}} \cdot G\left(\frac{\theta_l}{1-\kappa}\right) + \left(\frac{G_{\theta_l,2\alpha}(\theta \cdot (1-\kappa)) + G_{\theta_l,2\alpha}(\theta)}{2}\right) \cdot \left(1 - G\left(\frac{\theta_l}{1-\kappa}\right)\right)$$

$$G_{\theta_l'|10}(\theta) = \begin{cases} 1_{\{\theta \geq \theta_l\}} \cdot G(\theta_l \cdot (1-\kappa)) + \frac{G_{\theta_l \cdot (1-\kappa), 2\alpha}(\theta) + G_{\theta_l \cdot (1-\kappa), 2\alpha}(\theta \cdot (1-\kappa))}{2} \cdot (1 - G(\theta_l \cdot (1-\kappa))) & , \ \theta_l \cdot (1-\kappa) > \theta_{\min} \\ 1_{\{\theta \geq \theta_{\min}\}} \cdot G_{\theta_l \cdot (1-\kappa), \alpha}(\theta_{\min}) + \frac{G_{\theta_{\min}, 2\alpha}(\theta) + G_{\theta_{\min}, 2\alpha}(\theta \cdot (1-\kappa))}{2} \cdot \left(1 - G_{\theta_l \cdot (1-\kappa), \alpha}(\theta_{\min})\right) & , \ \theta_l \cdot (1-\kappa) < \theta_{\min} \end{cases}$$

When the state intervenes in the war of attrition by taking sides, it significantly alters the screening process inherent to the conflict. This intervention is not merely a matter of influencing the immediate outcome but also affects the reputational dynamics. The corollary shows that differential targeting by the state leads to an intriguing equilibrium consequence: a group that emerges victorious while being targeted by the state gains a higher reputation than it would have in the absence of such adversity. This enhanced reputation is a direct result of the perceived 'handicap' overcome by the group, signaling a higher level of strength and resilience. Therefore, the state's decision to target one group over another not only impacts the immediate strategic conflict behavior but also has future effects on the reputation of the incumbent in the second period.

Equilibrium conflict outcomes at $t = 1$ under $(r_I^{t=1}, r_C^{t=1}) \in \{(0,0),(1,1)\}$

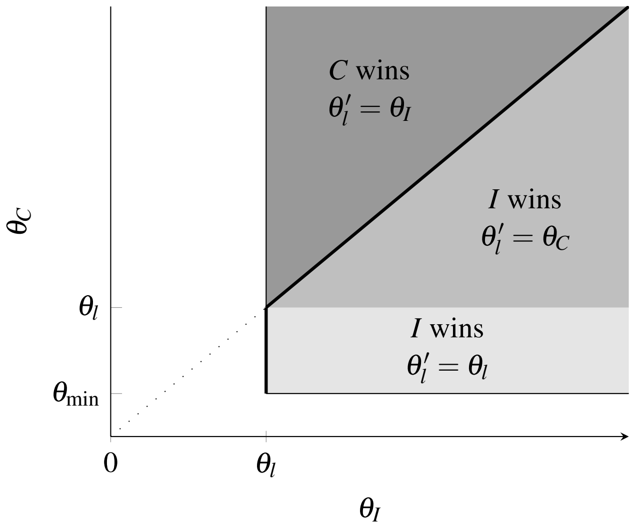

Figure 16: The graph shows equilibrium conflict outcomes at $t = 1$ regarding who wins and what reputation it is earned θ_l' when the state does not target any group or it targets both groups $(r_I^{t=1}, r_C^{t=1}) \in \{(0,0),(1,1)\}$. The graph was generated using $\theta_{\min} = 1$ and $\theta_l = 3$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$ and $\theta_l \geq \theta_{\min}$.

Equilibrium conflict outcomes at $t = 1$ under $(r_I^{t=1}, r_C^{t=1}) = (0, 1)$

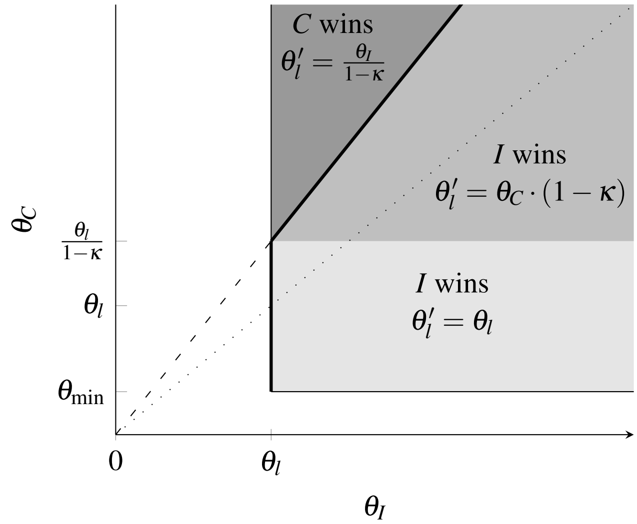

Figure 17: The graph shows equilibrium conflict outcomes at $t = 1$ regarding who wins and what reputation it is earned θ_l' when the state only targets the challenger $(r_I^{t=1}, r_C^{t=1}) = (0, 1)$. The graph was generated using $\theta_{\min} = 1$, $\theta_l = 3$ and $\kappa = \frac{1}{2}$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$, $\theta_l \geq \theta_{\min}$ and $\kappa \in (0, 1)$.

Equilibrium conflict outcomes at $t = 1$ under $(r_I^{t=1}, r_C^{t=1}) = (1,0)$, when $\theta_l \cdot (1-\kappa) > \theta_{\min}$

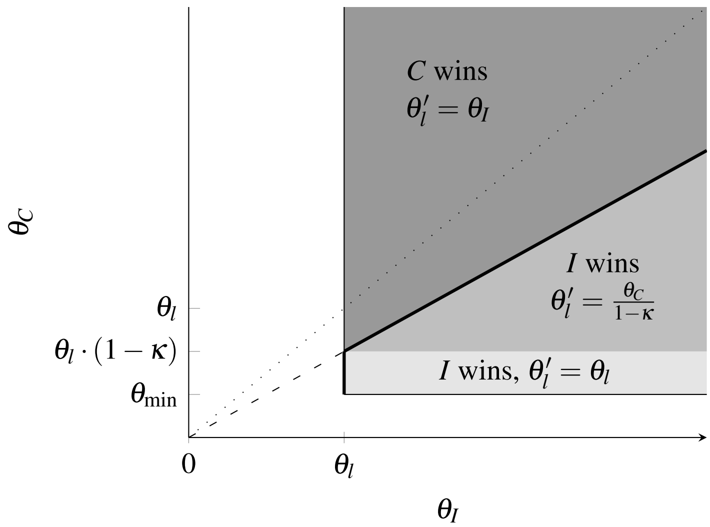

Figure 18: The graph shows equilibrium conflict outcomes at $t = 1$ regarding who wins and what reputation it is earned θ_l' when the state only targets the incumbent $(r_I^{t=1}, r_C^{t=1}) \in (1, 0)$ and the the incumbent has a strong reputation relative to the state and the challenger $\theta_l \cdot (1-\kappa) > \theta_{\min}$. The graph was generated using $\theta_{\min} = 1$, $\theta_l = 3$ and $\kappa = \frac{1}{3}$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$, $\theta_l \geq \theta_{\min}$ and $\kappa \in \left(0, 1 - \frac{\theta_{\min}}{\theta_l}\right)$.

Equilibrium conflict outcomes at $t = 1$ under $(r_I^{t=1}, r_C^{t=1}) = (1, 0)$, when $\theta_l \cdot (1-\kappa) < \theta_{\min}$

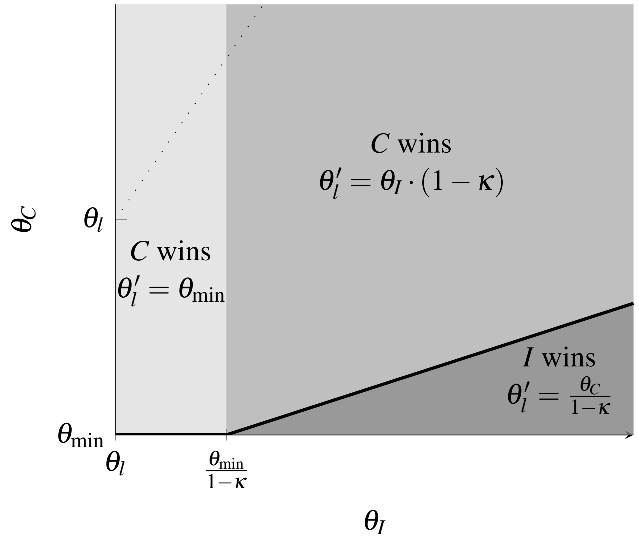

Figure 19: The graph shows equilibrium conflict outcomes at $t = 1$ regarding who wins and what reputation it is earned θ_l' when the state only targets the incumbent $(r_I^{t=1}, r_C^{t=1}) \in (1, 0)$ and the the incumbent has a weak reputation relative to the state and the challenger $\theta_l \cdot (1-\kappa) < \theta_{\min}$. The graph was generated using $\theta_{\min} = 1$, $\theta_l = 3$ and $\kappa = \frac{7}{9}$, but the qualitative properties of the graph do not change for different values of $\theta_{\min} > 0$, $\theta_l \geq \theta_{\min}$ and $\kappa \in \left(1 - \frac{\theta_{\min}}{\theta_l}, 1\right)$.

Corollary 3. *The cdfs of the equilibrium reputation of the incumbent θ_l' at $t = 2$ under different state targeting policies at $t = 1$ can be expressed as follows.*

$$G_{\theta_l'|00}(\theta) = G_{\theta_l'|11}(\theta) = \begin{cases} 1 - \frac{\theta_{\min}^{\alpha}\theta_l^{\alpha}}{\theta^{2\alpha}} & , \theta \geq \theta_l \\ 0 & , \theta < \theta_l \end{cases}$$

$$G_{\theta_l'|01}(\theta) = \begin{cases} 1 - \frac{(1-\kappa)^{\alpha} + (1-\kappa)^{-\alpha}}{2} \frac{\theta_{\min}^{\alpha}\theta_l^{\alpha}}{\theta^{2\alpha}} & , \theta > \frac{\theta_l}{1-\kappa} \\ 1 - (1-\kappa)^{\alpha} \left(\frac{\theta_{\min}}{\theta_l}\right)^{\alpha} \frac{\left(\frac{\theta_l}{\theta}\right)^{2\alpha} + 1}{2} & , \theta \in \left[\theta_l, \frac{\theta_l}{1-\kappa}\right) \\ 0 & . \theta < \theta_l \end{cases}$$

$$\theta_l \cdot (1-\kappa) > \theta_{\min} \Rightarrow G_{\theta_l'|01}(\theta) = \begin{cases} 1 - \frac{(1-\kappa)^{\alpha} + (1-\kappa)^{-\alpha}}{2} \frac{\theta_{\min}^{\alpha}\theta_l^{\alpha}}{\theta^{2\alpha}} & , \theta \geq \theta_l \\ 0 & , \theta < \theta_l \end{cases}$$

$$\theta_l \cdot (1-\kappa) < \theta_{\min} \Rightarrow G_{\theta_l'|01}(\theta) = \begin{cases} 1 - \frac{(1-\kappa)^{\alpha} + (1-\kappa)^{-\alpha}}{2} \frac{\theta_{\min}^{\alpha}\theta_l^{\alpha}}{\theta^{2\alpha}} & , \theta > \frac{\theta_{\min}}{1-\kappa} \\ 1 - (1-\kappa)^{\alpha} \left(\frac{\theta_l}{\theta_{\min}}\right)^{\alpha} \frac{\left(\frac{\theta_{\min}}{\theta}\right)^{2\alpha} + 1}{2} & , \theta \in \left[\theta_{\min}, \frac{\theta_{\min}}{1-\kappa}\right] \\ 0 & , \theta < \theta_{\min} \end{cases}$$

Where the following holds.

- $G_{\theta_l'|00}(\theta)$ *and* $G_{\theta_l'|01}(\theta)$ *are discontinuous only at* θ_l.
- *When* $\theta_l \cdot (1-\kappa) > \theta_{\min}$*,* $G_{\theta_l'|10}(\theta)$ *is discontinuous only at* θ_l.
- *When* $\theta_l \cdot (1-\kappa) < \theta_{\min}$*,* $G_{\theta_l'|10}(\theta)$ *is discontinuous only at* $\theta_{\min}$.
- $G_{\theta_l'|01}(\theta) = G_{\theta_l'|10}(\theta)$ *for all* $\theta \in \left[\frac{\theta_l}{1-\kappa}, \infty\right)$.
- *When* $\theta_l \cdot (1-\kappa) < \theta_{\min}$*,* $\theta_l'|10$ *first-order stochastically dominates* $\theta_l'|01$.
- *Except for the previous clause, there is no first-order stochastic dominance among* $\theta_l'|00$*,* $\theta_l'|01$ *and* $\theta_l'|10$.

The main insight from the aforementioned corollary is that the ex-ante distribution of the incumbent's reputation in the second period, as influenced by varying targeting policies, cannot generally be ranked through first-order stochastic dominance. This is crucial because it underscores that preferences for certain distributions of reputations hinge on assumptions regarding risk. Although our analysis presumes the state is risk-neutral, it's worth to acknowledging that altering assumptions about risk preferences—specifically, the state's risk tolerance towards violence—could significantly shift the findings on what constitutes optimal state policies.

Lemma 4. *Suppose the incumbent group has a reputation* $\theta_l \geq \theta_{\min}$ *at* $t = 1$*, then, in equilibrium, the expected value of the reputation at of the incumbent at* $t = 2$ *to the power of* $1-\alpha$ *after the state chooses targeting policy* $\left((r_I^{t=1}, r_C^{t=1}) = (r_I, r_C)\right)$*,* $\mathbb{E}\left[(\theta_l')^{1-\alpha}|r_I^{t=1}, r_C^{t=1}, \theta_l\right]$*, is given by the following.*

$$\mathbb{E}_{r_I r_C}\left[(\theta_l')^{1-\alpha}|\theta_l\right] =$$

$$\begin{cases} \left(G(\theta_l) + (1-G(\theta_l)) \cdot \frac{2\cdot\alpha}{3\cdot\alpha-1}\right) \cdot \theta_l^{1-\alpha} & , (r_I, r_C) \in \{(0,0),(1,1)\} \\ \left(G\left(\frac{\theta_l}{1-\kappa}\right) + \left(1 - G\left(\frac{\theta_l}{1-\kappa}\right)\right) \cdot \frac{2\cdot\alpha}{3\cdot\alpha-1} \cdot \frac{(1-\kappa)^{1-\alpha}+1}{2}\right) \cdot \theta_l^{1-\alpha} & , (r_I, r_C) = (0,1) \\ \left(G(\theta_l \cdot (1-\kappa)) + (1-G(\theta_l \cdot (1-\kappa))) \cdot \frac{2\cdot\alpha}{3\cdot\alpha-1} \cdot \frac{(1-\kappa)^{1-\alpha}+1}{2}\right) \cdot \theta_l^{1-\alpha} & , (r_I, r_C) = (1,0) \text{ and } \theta_l \cdot (1-\kappa) > \theta_{\min} \\ \left(G_{\theta_l\cdot(1-\kappa),\alpha}(\theta_{\min}) + \left(1 - G_{\theta_l\cdot(1-\kappa),\alpha}(\theta_{\min})\right) \cdot \frac{2\cdot\alpha}{3\cdot\alpha-1} \cdot \frac{(1-\kappa)^{-(1-\alpha)}+1}{2}\right) \cdot \theta_{\min}^{1-\alpha} & , (r_I, r_C) = (1,0) \text{ and } \theta_l \cdot (1-\kappa) < \theta_{\min} \end{cases}$$

Despite the assumption that the state is risk-neutral regarding violence, the state's induced preferences over reputation are nonlinear. Understanding and computing these nuanced preferences over reputation is essential for delineating the optimal policy for the state, particularly as it navigates the trade-off between the immediate and long-term repercussions of its interventions.

Proposition 4. *If targeting groups is costless for the state, $\beta = \gamma = \lambda = 0$, then the optimal targeting policy for the state at $t = 1$ is given by the following.*

$$\left(r_I^{*^{t=1}}\left[\theta_l'\right], r_C^{*^{t=1}}\left[\theta_l'\right]\right) = \begin{cases} (1,1) & , \alpha \in \left(\frac{1}{2}, 1\right) \\ \{(0,1),(1,1)\} & , \alpha = 1 \\ (1,0) & , \alpha \in (1,\infty) \text{ and } \theta_l < \theta_l^* \\ (0,1) & , \alpha \in (1,\infty) \text{ and } \theta_l > \theta_l^* \end{cases}$$

Where $\theta_l^* > \theta_{\min}$.

Even in scenarios where targeting only the incumbent does not efficiently reduce immediate violence, this strategy may emerge as the most effective policy from a broader perspective. This counterintuitive outcome hinges on the fact that targeting the incumbent can lead to a higher expected continuation reputation for the incumbent. In contexts where the state values high reputations among incumbents—due to their conflict-deterrent effects—this approach strategically leverages the long-term benefits of enhancing an incumbent's reputation to suppress potential future conflicts. Consequently, the policy's efficacy is not solely measured by its immediate impact on violence but also by its influence on the strategic calculus of conflict across periods.

Thus, when the state prioritizes the minimization of expected violence over multiple periods, it may opt for policies that initially seem less efficient in reducing violence. By targeting only the incumbent and thereby boosting its reputation, the state exploits the deterrent effect of a strong incumbent reputation to achieve a more peaceful equilibrium in the long run. This strategic choice underscores the complexity of state interventions in conflicts, where the indirect effects on reputational dynamics play a critical role in shaping the optimal policy framework. The state's preference for high-reputation incumbents, in this case, acts as a deliberate strategy to harness reputational deterrence as a means to curtail violence over an extended horizon.

Discussion

This paper introduces a dynamic framework to explore the interplay between the state and violent non-state actors, illuminating the complexity of targeting policies in reducing violence. It hinges on the premise that the state's primary role, as posited in classical social contract theory, is to provide security—a principle complicated by the reality of violent non-state actors vying for control and resources. Through a strategic interaction lens, the model delves into the sequential nature of conflicts, the impact of state interventions on these actors, and the strategic calculations underpinning the distribution of violence.

The core of the model lies in its portrayal of conflicts as sequential Wars of Attrition under asymmetric information, with non-state groups engaging in violent contests over rents. This approach underscores the inherent negative externalities of such conflicts on state interests, driving the state's interventions aimed at minimizing violence. However, the analysis reveals nuanced strategic considerations: the state's interventions, while aimed at dampening the intensity and likelihood of violence, must also navigate the reputational dynamics of non-state actors, influenced significantly by the outcomes of these interventions.

Central to the model's findings is the role of public information regarding the strength of conflict winners in mediating future conflicts. This aspect introduces a trade-off between deterring violence through reputational mechanisms and the potential for intensified conflicts driven by the emergence of stronger challengers. Interestingly, the model suggests that the state's optimal targeting policy is not uniformly directed at all actors; rather, it varies with the state's capacity and the strategic landscape dictated by the non-state actors' reputations.

Furthermore, the analysis extends to the strategic choices available to the state in the initial period of conflict, emphasizing the long-term implications of these choices on violence and the strategic positioning of non-state actors. Here, the state's calculus involves not just immediate violence minimization but also the shaping of future conflict dynamics through reputational effects—a reflection of the intricate balance between immediate security concerns and the long-term strategic landscape of state-non-state interactions.

In essence, the model offers a rich analytical framework that captures the multifaceted nature of state interventions in contexts of violent non-state competition. It sheds light on the strategic underpinnings of violence distribution and the critical role of reputational dynamics in shaping both state policies and the behavior of non-state actors, offering nuanced insights into the complexities of ensuring security in the face of persistent and organized violence.

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Appendix of proofs

Appendix

Proof of lemma 1. Sketch of proof:

First let's show that the support of the primitive distribution of *types* must be unbounded. Note that because $Supp[\theta] = [\theta_l, \theta_{\max})$, then $Supp[\theta \cdot (1-\kappa)] = [\theta_l \cdot (1-\kappa), \theta_{\max} \cdot (1-\kappa))$ while $Supp[\theta | \{\theta > \theta_l \cdot (1-\kappa)\}] = [\theta_l \cdot (1-\kappa), \theta_{\max})$ which implies that $\theta_{\max} = \infty$.

Next note that when $\theta_l \geq \theta_{\min}$ then the cdf of $\Theta | \{\Theta > \theta_l\}$ is

$$G(\theta|\theta_l) = \frac{G(\theta) - G(\theta_l \cdot (1-\kappa))}{1 - G(\theta_l \cdot (1-\kappa))}$$

Next note that the cdf of $(1-\kappa) \cdot \Theta | \{\Theta > \theta_l\}$ is given by

$$\frac{G\left(\frac{\theta}{1-\kappa}\right) - G(\theta_l)}{1 - G(\theta_l)}$$

Defining $\bar{G}[\theta] := 1 - G(\theta)$, and re-labeling $x := 1-\kappa$ then we have

$$\bar{G}(\theta) \cdot \bar{G}(\theta') = \bar{G}\left(\frac{\theta}{x}\right) \cdot \bar{G}(\theta' \cdot x)$$
$$\iff \ln \bar{G}(\theta) + \ln \bar{G}(\theta') = \ln \bar{G}\left(\frac{\theta}{x}\right) + \ln \bar{G}(\theta' \cdot x)$$

Taking the derivative with respect to θ' we get

$$\left(\ln \bar{G}\right)'(\theta' x) = \frac{\left(\ln \bar{G}\right)'(\theta')}{x}$$

Integrating with respect to x we get that

$$\frac{\ln \bar{G}(\theta' x)}{\theta'} = \left(\ln \bar{G}\right)'(\theta') \cdot \ln(x) + c_0$$
$$\iff \bar{G}(\theta' x) = x^{\left(\ln \bar{G}\right)'(\theta') \cdot \theta'} \cdot \exp(c_0)$$

Which implies that F must have the functional form of the cdf of a *Pareto* distribution.

□

Proof of proposition 1. To ease notation, let's relabel θ_l' as θ_l.

Now let's first prove that $(\sigma[\theta_l], \sigma[\theta_l])$ is a Bayesian equilibrium. First note that because the period-t payoffs in the absence of any targeting from the state for the groups can be rearranged as:

$$u_j^t = \zeta_j \cdot \left(\frac{\nu_j}{\zeta_j} \cdot 1_{\{s_j^t > s_{-j}^t\}} - \min\{s_j^t, s_{-j}^t\} \right)$$

And because the optimal quitting points are characterized by the first-order conditions (Fudenberg and Tirole 1986), then from here on we will just refer to $\theta_j := \frac{\nu_j}{\zeta_j}$ as "the *type*" for group j, and in equilibrium this quantity will determine the stopping point of any group j in the second period t in the absence of any targeting from the state.

Note that in the case where both side's *types* independently distribute as $\frac{\nu_j}{\zeta_j} \sim G(\cdot|\theta_l)$ it is known that the strategy τ of each player in the symmetric equilibrium is characterized by the following ordinary differential equation (Fudenberg and Tirole 1991, example 6.3)

$$\begin{aligned}
\tau'(\theta) &= \theta \cdot \frac{G(\theta|\theta_l)}{1 - G(\theta|\theta_l)} && , \ \theta \in (\theta_l, \infty) \\
&= \theta \cdot \frac{G(\theta)}{1 - G(\theta)} && , \ \theta \in (\theta_l, \infty) \\
&= \alpha && , \ \theta \in (\theta_l, \infty)
\end{aligned}$$

Where τ must be continuous and differentiable. By the fundamental theorem of calculus then we have

$$\tau(\theta) = \tau(\theta_l) + \underbrace{\int_{\theta_l}^{\theta} x \cdot \frac{G'(x)}{1 - G(x)} dx}_{\alpha \cdot (\theta - \theta_l)}, \ \theta \in (\theta_l, \infty)$$

Where it must be the case that the lowest most *type* in the support has an immediate quitting point $\tau(\theta_l) = 0$.

Now we will show that if a challenger whose type is below that faces an incumbent whose *type* distributes as $\theta_I \sim G(\cdot|\theta_l)$ that follows strategy τ, then all the challengers with *type* below θ_l, $\theta_C < \theta_l$, will have an immediate quitting point as a best response. This is because

$$s_C \mapsto \mathbb{E}\left[u_C(s_C, \tau(\theta_I), \theta_C) | \theta_C\right]$$

is strictly decreasing for all stopping points $s_C \in [0, \infty)$ when $\theta_C < \theta_l$. By symmetry $\tau(\theta_C)$ is also the best response of a challenger with *type* $\theta_C \geq \theta_l$ to an incumbent whose *type* distributes as $\theta_I \sim G(\cdot|\theta_l)$ and follows strategy τ, and from this it is straightforward that no incumbent will want to deviate from following τ when facing challengers whose *types* distribute as $\theta_C \sim G(\cdot)$ and follow strategy $\sigma[\theta_l]$. This proves that $(\sigma[\theta_l], \sigma[\theta_l])$ is a Bayesian equilibrium.

To prove that the equilibrium is unique we will proceed by contradiction. Suppose there is another equilibrium of this subgame, and denote it as $\iota_I(\theta_I), \iota_C(\theta_C)$, from standard arguments we know that (i) ι_I and ι_C must be continuous and non-decreasing (Fudenberg and Tirole 1991), at most one player concedes immediately with positive probability (Hendricks et al. 1988; Amann and Leininger 1996) and the strategies are differentiable except for the highest-most type that concedes immediately (Fudenberg and Tirole 1991; Abreu and Gul 2000). W.l.o.g. suppose that only challenger types between $\left(\theta_{\min}, \bar{\theta}\right)$, with $\bar{\theta} < \theta_l$, concede immediately: $\iota_C^{-1}(0) = \left(\theta_{\min}, \bar{\theta}\right)$. Defining $\Phi_I := \iota_I^{-1}$ and $\Phi_C := \left(\iota_{I|\left(\bar{\theta},\infty\right)}\right)^{-1}$, then from Fudenberg and Tirole (1991) we know that Φ_I and Φ_C are increasing, differentiable and that it must be the case that

$$
\begin{aligned}
\Phi_I'(s) &= \frac{1}{\alpha} \cdot \frac{\Phi_I(s)}{\Phi_C(s)} \\
\Phi_C'(s) &= \frac{1}{\alpha} \cdot \frac{\Phi_C(s)}{\Phi_I(s)} \\
\Phi_I(0) &= \theta_l \\
\Phi_C(0) &= \bar{\theta}
\end{aligned}
$$

The system of ordinary differential equations with border conditions only admits the following as a solution.

$$
\begin{aligned}
\Phi_I(s) &= \theta_l \cdot \frac{\theta_l \cdot \exp\left(\frac{s}{\alpha} \cdot \frac{\theta_l - \bar{\theta}}{\theta_l \cdot \bar{\theta}}\right) - \bar{\theta}}{\theta_l - \bar{\theta}} \\
\Phi_C(s) &= \bar{\theta} \cdot \frac{\theta_l - \bar{\theta} \cdot \exp\left(-\frac{s}{\alpha} \cdot \frac{\theta_l - \bar{\theta}}{\theta_l \cdot \bar{\theta}}\right)}{\theta_l - \bar{\theta}}
\end{aligned}
$$

Then noting that $\lim_{s\to\infty} \Phi_C(s) = \frac{\theta_l \cdot \bar{\theta}}{\theta_l - \bar{\theta}}$ which yield s a contradiction with the fact that $\iota_{C|\left(\bar{\theta},\infty\right)}$ is an increasing function from $\left(\bar{\theta}, \infty\right)$ to $[0, \infty)$. Thus there is no other equilibrium of this subgame apart from (σ, σ).

Next let's compute the extensive and intensive margin, and the expected violence.

The extensive margin is given by the following

$$\begin{aligned}
EM_{00}^{t=2}[\theta_l] &= Pr[\min\{\sigma[\theta_l](\theta_I), \sigma[\theta_l](\theta_C)\} > 0] \\
&= Pr[\sigma[\theta_l](\theta_C) > 0] \\
&= Pr[\theta_C \geq \theta_l] \\
&= 1 - G(\theta_l) \\
&= \left(\frac{\theta_{\min}}{\theta_l}\right)^{\alpha}
\end{aligned}$$

Which is clearly decreasing in θ_l.

The intensive margin is given by the following

$$\begin{aligned}
IM_{00}^{t=2}[\theta_l] &= \mathbb{E}[v[\theta_l] | v[\theta_l] > 0] \\
&= \mathbb{E}[\min\{\sigma[\theta_l](\theta_I), \sigma[\theta_l](\theta_C)\} | \theta_C \geq \theta_l]
\end{aligned}$$

From the *Revenue Equivalence Theorem* (Myerson 1981) we know that

$$\mathbb{E}[2 \cdot \min\{\sigma[\theta_l](\theta_I), \sigma[\theta_l](\theta_C)\} | \theta_C \geq \theta_l] = \mathbb{E}[\min\{\theta_I, \theta_C\} | \theta_C \geq \theta_l]$$

Thus

$$\begin{aligned}
IM_{00}^{t=2}[\theta_l] &= \mathbb{E}[v[\theta_l] | v[\theta_l] > 0] \\
&= \frac{\mathbb{E}[\min\{\theta_I, \theta_C\} | \theta_C \geq \theta_l]}{2} \\
&= \frac{\alpha}{2 \cdot \alpha - 1} \cdot \theta_l
\end{aligned}$$

Thus $IM_{00}^{t=2}[\theta_l]$ is increasing in θ_l.

The expected violence is given by

$$EV_{00}^{t=2}[\theta_l] = \mathbb{E}[\min\{\theta_I, \theta_C\} | \theta_C \geq \theta_l'] \cdot \frac{1 - G(\theta_l)}{2}$$
$$= \frac{\alpha \cdot \theta_{\min}^{\alpha}}{2 \cdot \alpha - 1} \cdot \theta_l^{1-\alpha}$$

Where $EV_{00}^{t=2}[\theta_l]$ is increasing if and only if $\alpha \in \left(\frac{1}{2}, 1\right)$, constant if and only if $\alpha = 1$ and decreasing if and only if $\alpha > 1$.

□

Proof of proposition 2. To ease notation, let's relabel θ_l' as θ_l.

First note that when the state chose not to target any group, we know from proposition 1 that the equilibrium is given by $(\sigma[\theta_l], \sigma[\theta_l])$ and that the extensive margin, intensive margin and the expected violence are provided by proposition 1 as well.

Now suppose the state only targets the challenger $(r_I, r_C) = (0, 1)$. Note that this case is isomorphic to a case where the state does not target any group and the scale parameter of the distribution of the type of the challenger is $\theta_{\min} \cdot (1 - \kappa)$. From proposition 1, the equilibrium strategies are given by

$$\sigma_I[0, 1, \theta_l](\theta_I) = \sigma[\theta_l](\theta_I)$$
$$\sigma_C[0, 1, \theta_l](\theta_C) = \sigma[\theta_l]\left(\theta_C \cdot (1 - \kappa)\right)$$

The intensive margin is the exact same as if there was no intervention, $IM_{01}^{t=2}[\theta_l] = \frac{\alpha}{2 \cdot \alpha - 1} \cdot \theta_l$, but the extensive margin and the expected violence are given by the following.

$$EM_{01}^{t=2}[\theta_l] = Pr[\min\{\sigma_I[0, 1, \theta_l](\theta_I), \sigma_C[0, 1, \theta_l](\theta_C)\} > 0]$$
$$= Pr[\sigma[0, 1, \theta_l](\theta_C) > 0]$$
$$= Pr[\sigma[\theta_l](\theta_C \cdot (1 - \kappa)) > 0]$$
$$= Pr[\theta_C \cdot (1 - \kappa) \geq \theta_l]$$
$$= 1 - G\left(\frac{\theta_l}{1 - \kappa}\right)$$
$$= (1 - \kappa)^{\alpha} \left(\frac{\theta_{\min}}{\theta_l}\right)^{\alpha}$$

Then

$$EV_{01}^{t=2}[\theta_l] = (1-\kappa)^{\alpha} \cdot \frac{\alpha}{2\alpha - 1} \theta_{\min}^{\alpha} {\theta_l}^{1-\alpha}$$

Now suppose the state targets both groups $(r_I, r_C) = (1, 1)$. Note that this case is isomorphic to a case where the state does not target any group, the reputation of the incumbent is given by $\theta_l \cdot (1-\kappa)$ and and the scale parameter of the distribution of the type of the challenger is $\theta_{\min} \cdot (1-\kappa)$. From proposition 1 then the equilibrium strategies are given by

$$\sigma_I[1, 1, \theta_l](\theta_I) = \sigma[\theta_l \cdot (1-\kappa)](\theta_I \cdot (1-\kappa))$$
$$\sigma_C[1, 1, \theta_l](\theta_C) = \sigma[\theta_l \cdot (1-\kappa)](\theta_C \cdot (1-\kappa))$$

The extensive margin is the same as in the case with no targeting, $EM_{11}^{t=2}[\theta_l] = \left(\frac{\theta_{\min}}{\theta_l}\right)^{\alpha}$, and the intensive margin and the expected violence are given by the following.

$$IM_{11}^{t=2}[\theta_l] = (1-\kappa) \cdot \frac{\alpha}{2 \cdot \alpha - 1} \cdot \theta_l$$
$$EV_{11}^{t=2}[\theta_l] = (1-\kappa) \cdot \frac{\alpha}{2\alpha - 1} \theta_{\min}^{\alpha} {\theta_l}^{1-\alpha}$$

Now suppose the state only targets the incumbent $(r_I, r_C) = (1, 0)$. Let's separate the analysis in two cases:

- Case $\theta_l \cdot (1-\kappa) > \theta_{\min}$: Note that this case is isomorphic to a case where the state does not target any group and the reputation of the incumbent is given by $\theta_l \cdot (1-\kappa)$. From proposition 1 then the equilibrium strategies are given by

$$\sigma_I[1, 0, \theta_l](\theta_I) = \sigma[\theta_l \cdot (1-\kappa)](\theta_I \cdot (1-\kappa))$$
$$\sigma_C[1, 0, \theta_l](\theta_C) = \sigma[\theta_l \cdot (1-\kappa)](\theta_C)$$

  The extensive margin is given by the following

$$
\begin{aligned}
EM_{10}^{t=2}[\theta_l] &= Pr[\min\{\sigma_I[1,0,\theta_l](\theta_I), \sigma_C[1,0,\theta_l](\theta_C)\} > 0] \\
&= Pr[\sigma[1,0,\theta_l](\theta_C) > 0] \\
&= Pr[\sigma[\theta_l \cdot (1-\kappa)](\theta_C) > 0] \\
&= Pr[\theta_C \geq \theta_l \cdot (1-\kappa)] \\
&= 1 - F(\theta_l \cdot (1-\kappa)) \\
&= (1-\kappa)^{-\alpha} \left(\frac{\theta_{\min}}{\theta_l} \right)^{\alpha}
\end{aligned}
$$

The intensive margin and the expected violence are given by the following

$$
\begin{aligned}
IM_{10}^{t=2}[\theta_l] &= (1-\kappa) \cdot \frac{\alpha}{2 \cdot \alpha - 1} \cdot \theta_l \\
EV_{10}^{t=2}[\theta_l] &= (1-\kappa)^{1-\alpha} \cdot \frac{\alpha}{2 \cdot \alpha - 1} \cdot \theta_{\min}^{\alpha} \theta_l^{1-\alpha}
\end{aligned}
$$

- Case $\theta_l \cdot (1-\kappa) < \theta_{\min}$: Note that this case is isomorphic to a case where the state does not target any group, and the roles of incumbent and challenger are flipped: the incumbent "becomes the challenger" with a scale such that the distribution of his type has a scale parameter of $\theta_l \cdot (1-\kappa)$; the challenger "becomes the incumbent" with a reputation of $\theta_{\min}$. From proposition 1, then the equilibrium strategies are given by

$$
\begin{aligned}
\sigma_I[1,0,\theta_l](\theta_I) &= \sigma[\theta_{\min}](\theta_I \cdot (1-\kappa)) \\
\sigma_C[1,0,\theta_l](\theta_C) &= \sigma[\theta_{\min}](\theta_C)
\end{aligned}
$$

The extensive and intensive margin and the expected violence are given by the following

$$EM_{10}^{t=2}[\theta_l'] = (1-\kappa)^\alpha \cdot \left(\frac{\theta_l}{\theta_{\min}}\right)^\alpha$$
$$IM_{10}^{t=2}[\theta_l'] = \frac{\alpha}{2\cdot\alpha-1}\cdot\theta_{\min}$$
$$EV_{10}^{t=2}[\theta_l'] = (1-\kappa)^\alpha \cdot \frac{\alpha}{2\cdot\alpha-1}\cdot\theta_{\min}^{1-\alpha}\theta_l^\alpha$$

Finally note that the expected violence can be expressed as follows.

$$EV_{r_I r_C}^{t=2}[\theta_l'] = \begin{cases} \frac{\alpha}{2\alpha-1}\theta_{\min}^\alpha {\theta_l'}^{1-\alpha} & , (r_I, r_C) = (0,0) \\ (1-\kappa)^\alpha \cdot \frac{\alpha}{2\alpha-1}\theta_{\min}^\alpha {\theta_l'}^{1-\alpha} & , (r_I, r_C) = (0,1) \\ (1-\kappa)^{1-\alpha} \cdot \frac{\alpha}{2\alpha-1}\theta_{\min}^\alpha {\theta_l'}^{1-\alpha} & , (r_I, r_C) = (1,0) \text{ and } \theta_l' \cdot (1-\kappa) > \theta_{\min} \\ (1-\kappa)^\alpha \cdot \frac{\alpha}{2\alpha-1}\theta_{\min}^{1-\alpha} {\theta_l'}^{\alpha} & , (r_I, r_C) = (1,0) \text{ and } \theta_l' \cdot (1-\kappa) < \theta_{\min} \\ (1-\kappa) \cdot \frac{\alpha}{2\alpha-1}\theta_{\min}^\alpha {\theta_l'}^{1-\alpha} & , (r_I, r_C) = (1,1) \end{cases}$$

Note that for any $\theta_l' \geq \theta_{\min}$, $EV_{00}^{t=2}[\theta_l'] > EV_{11}^{t=2}[\theta_l'], EV_{01}^{t=2}[\theta_l']$ thus $(r_I, r_C) = (0,0)$ does not yield the lowest expected violence.

Let's show that when $\alpha < 1$, $(r_I, r_C) = (1,1)$ is the policy that induces the lowest expected violence. Because $(1-\kappa)^\alpha > 1-\kappa$, then $EV_{11}^{t=2}[\theta_l'] < EV_{01}^{t=2}[\theta_l']$. By the same argument $EV_{11}^{t=2}[\theta_l'] < EV_{10}^{t=2}[\theta_l']$ when $\theta_l' \cdot (1-\kappa) > \theta_{\min}$. The claim that $EV_{11}^{t=2}[\theta_l'] < EV_{10}^{t=2}[\theta_l']$ when $\theta_l' \cdot (1-\kappa) < \theta_{\min}$ follows from the observation that $EV_{11}^{t=2}[\theta_{\min}] < EV_{10}^{t=2}[\theta_{\min}]$ and $\frac{\partial EV_{11}^{t=2}}{\partial \theta_l'} < \frac{\partial EV_{10}^{t=2}}{\partial \theta_l'}$.

Finally let's show that when $\alpha > 1$, $(r_I, r_C) = (0,1)$ is the policy that induces the lowest expected violence. Because $(1-\kappa)^\alpha < 1-\kappa$, then $EV_{11}^{t=2}[\theta_l'] > EV_{01}^{t=2}[\theta_l']$. By the same argument $EV_{01}^{t=2}[\theta_l'] < EV_{10}^{t=2}[\theta_l']$ when $\theta_l' \cdot (1-\kappa) > \theta_{\min}$. The claim that $EV_{01}^{t=2}[\theta_l'] < EV_{10}^{t=2}[\theta_l']$ when $\theta_l' \cdot (1-\kappa) < \theta_{\min}$ follows from the observation that $EV_{01}^{t=2}[\theta_{\min}] = EV_{10}^{t=2}[\theta_{\min}]$ and $\frac{\partial EV_{01}^{t=2}}{\partial \theta_l'} < 0 < \frac{\partial EV_{10}^{t=2}}{\partial \theta_l'}$. □

Proof of lemma 2. First denote θ_l' as θ_l.

First re-write the expected utility of the state under policies $(0,1)$ and $(1,0)$.

$$\begin{aligned}
\mathbb{E}[u_S^{t=2}|(r_I,r_C),\theta_l] &= -EV[r_I,r_C,\theta_l]\cdot(\Gamma(r_I+r_C)\cdot\beta+1)-1_{\{r_I+r_C>0\}}\cdot\gamma \\
&= \begin{cases} -EV[0,1,\theta_l]\cdot(\beta+1)-\gamma & ,\ (r_I,r_C)=(0,1) \\ -EV[1,0,\theta_l]\cdot(\beta+1)-\gamma & ,\ (r_I,r_C)=(1,0) \end{cases} \\
&= -\begin{cases} F\left[\theta_l,(1-\kappa)\cdot\theta_{\min}\right]\cdot(\beta+1)+\gamma & ,\ (r_I,r_C)=(0,1) \\ F\left[(1-\kappa)\cdot\theta_l,\theta_{\min}\right]\cdot(\beta+1)+\gamma & ,\ (r_I,r_C)=(1,0) \end{cases}
\end{aligned}$$

Then $\mathbb{E}[u_S^{t=2}|(0,1),\theta_l] > \mathbb{E}[u_S^{t=2}|(1,0),\theta_l]$ if and only if $F\left[\theta_l,(1-\kappa)\cdot\theta_{\min}\right] < F\left[(1-\kappa)\cdot\theta_l,\theta_{\min}\right]$. Next let's separate the analysis in two cases.

- <u>Case $\theta_l\cdot(1-\kappa)>\theta_{\min}$:</u>

$$\begin{aligned}
F\left[\theta_l,(1-\kappa)\cdot\theta_{\min}\right] < F\left[(1-\kappa)\cdot\theta_l,\theta_{\min}\right] &\iff (1-\kappa)^{\alpha}\cdot F\left[\theta_l,\theta_{\min}\right] < (1-\kappa)^{1-\alpha}\cdot F\left[\theta_l,\theta_{\min}\right] \\
&\iff (1-\kappa)^{\alpha} < (1-\kappa)^{1-\alpha} \\
&\iff 1-\alpha<\alpha \\
&\iff \frac{1}{2}<\alpha
\end{aligned}$$

  Which is true.

- <u>Case $\theta_l\cdot(1-\kappa)<\theta_{\min}$:</u>

$$\begin{aligned}
F\left[\theta_l,(1-\kappa)\cdot\theta_{\min}\right] < F\left[(1-\kappa)\cdot\theta_l,\theta_{\min}\right] &\iff (1-\kappa)^{\alpha}\cdot\theta_{\min}^{\alpha}\cdot\theta_l^{1-\alpha} < (1-\kappa)^{\alpha}\cdot\theta_l^{\alpha}\cdot\theta_{\min}^{1-\alpha} \\
&\iff \theta_{\min}^{2\cdot\alpha-1} < \theta_l^{2\cdot\alpha-1} \\
&\iff \theta_{\min}<\theta_l
\end{aligned}$$

  Which is true.

□

Proof of proposition 3. First denote θ_l' as θ_l. Next, note that by lemma 2 we know that targeting only the incumbent cannot be an optimal policy for the state at $t=2$.

Note that we can express the expected period-2 payoff for the state as a function of the policy (r_I, r_C).

$$\mathbb{E}[u_S^2|(r_I,r_C),\theta_l] = -\begin{cases} F[\theta_l,\theta_{\min}] & ,\ (r_I,r_C)=(0,0) \\ (1-\kappa)^\alpha \cdot F[\theta_l,\theta_{\min}]\cdot(\beta+1)+\gamma & ,\ (r_I,r_C)=(0,1) \\ (1-\kappa)\cdot F[\theta_l,\theta_{\min}]\cdot(\lambda\cdot\beta+1)+\gamma & ,\ (r_I,r_C)=(1,1) \end{cases}$$

Next let's compare the expected payoffs of each policy

- The state prefers targeting no group over targeting only the challenger if and only if

$$\begin{aligned} \mathbb{E}[u_S^2|(r_I,r_C)=(0,0),\theta_l] > \mathbb{E}[u_S^2|(r_I,r_C)=(0,1),\theta_l] &\iff F[\theta_l,\theta_{\min}] < (1-\kappa)^\alpha F[\theta_l,\theta_{\min}]\cdot(\beta+1)+\gamma \\ &\iff (1-\kappa)^\alpha(\beta+1) > 1 \\ &\quad \vee F[\theta_l,\theta_{\min}] < \frac{\gamma}{1-(1-\kappa)^\alpha(\beta+1)} \end{aligned}$$

- The state prefers targeting no group over targeting both if and only if

$$\begin{aligned} \mathbb{E}[u_S^2|(r_I,r_C)=(0,0),\theta_l] > \mathbb{E}[u_S^2|(r_I,r_C)=(1,1),\theta_l] &\iff F[\theta_l,\theta_{\min}] < (1-\kappa)F[\theta_l,\theta_{\min}]\cdot(\lambda\beta+1)+\gamma \\ &\iff [(1-\kappa)(\lambda\beta+1) > 1] \\ &\quad \vee \left[F[\theta_l,\theta_{\min}] < \frac{\gamma}{1-(1-\kappa)(\lambda\beta+1)}\right] \end{aligned}$$

- The state prefers targeting only the challenger over targeting both if and only if

$$\begin{aligned} \mathbb{E}[u_S^2|(r_I,r_C)=(0,1),\theta_l] > \mathbb{E}[u_S^2|(r_I,r_C)=(1,1),\theta_l] &\iff (1-\kappa)^\alpha F[\theta_l,\theta_{\min}]\cdot(\beta+1)+\gamma \\ &\quad < (1-\kappa)F[\theta_l,\theta_{\min}]\cdot(\lambda\beta+1)+\gamma \\ &\iff [\alpha \geq 1] \vee \left[\kappa < 1 - \sqrt[1-\alpha]{\frac{\beta+1}{\lambda\beta+1}}\right] \end{aligned}$$

□

Proof of lemma 3. First note that when the state is targeting both or none of the groups the ex-ante distribution of the reputation of the incumbent in the second period $t = 2$ is the same and given by the following.

$$\theta_l' | \{r_I^{t=1} = r_C^{t=1}\} \sim \min\{\theta_I, \max\{\theta_C, \theta_l\}\}$$

Suppose at $t = 1$ the state was targeting only the challenger C. Then if the challenger wins it will have a reputation of $\theta_l' = \frac{\theta_I}{1-\kappa}$ and if it loses then the incumbent will have a reputation of $\theta_l' = \max\{\theta_l, \theta_C \cdot (1-\kappa)\}$. Thus the ex-ante distribution of the reputation of the incumbent at $t = 2$ when the state targets only the challenger is given by the following.

$$\theta_l' | \{(r_I^{t=1}, r_C^{t=1}) = (0,1)\} \sim \begin{cases} \frac{\theta_I}{1-\kappa} & , \theta_C \cdot (1-\kappa) > \theta_I \\ \max\{\theta_l, \theta_C \cdot (1-\kappa)\} & , \theta_C \cdot (1-\kappa) < \theta_I \end{cases}$$

Suppose at $t = 1$ the state was targeting only the incumbent I. Then if the challenger wins it will have a reputation of $\theta_l' = \max\{\theta_{\min}, \theta_I \cdot (1-\kappa)\}$ and if it loses then the incumbent at $t = 2$ will have a reputation of $\theta_l' = \max\left\{\theta_l, \frac{\theta_C}{1-\kappa}\right\}$. Thus the ex-ante distribution of the reputation of the incumbent at $t = 2$ when the state targets only the challenger is given by the following.

$$\theta_l' | \{(r_I^{t=1}, r_C^{t=1}) = (1,0)\} \sim \begin{cases} \max\{\theta_{\min}, \theta_I \cdot (1-\kappa)\} & , \theta_C \cdot (1-\kappa) > \theta_I \\ \max\left\{\theta_l, \frac{\theta_C}{1-\kappa}\right\} & , \theta_C \cdot (1-\kappa) < \theta_I \end{cases}$$

Now note that $\min\{\theta_I, \theta_C\} | \{\theta_C > \theta_l\} \sim Pareto(\theta_l, 2 \cdot \alpha)$, thus

$$Pr[\theta_l' \leq \theta | r_I^{t=1} = r_C^{t=1}] = \begin{cases} 0 & , \theta < \theta_l \\ G(\theta_l) + (1 - G(\theta_l)) \cdot \left(1 - \left(\frac{\theta_l}{\theta}\right)^{2\cdot\alpha}\right) & , \theta > \theta_l \end{cases}$$

Next denote $P_{r_I^{t=1} r_C^{t=1}}[\theta_l' \leq \theta] := Pr[\theta_l' \leq \theta | (r_I^{t=1}, r_C^{t=1})]$

$$\begin{aligned} P_{01}[\theta_l' \leq \theta] &= P_{01}[\theta_l' \leq \theta | \theta_C \cdot (1-\kappa) < \theta_l] \cdot P_{01}[\theta_C \cdot (1-\kappa) < \theta_l] + P_{01}[\theta_l' \leq \theta | \theta_C \cdot (1-\kappa) > \theta_l] \cdot P_{01}[\theta_C \cdot (1-\kappa) > \theta_l] \\ &= 1_{\{\theta \geq \theta_l\}} \cdot G\left(\frac{\theta_l}{1-\kappa}\right) + P_{01}[\theta_l' \leq \theta | \theta_C \cdot (1-\kappa) > \theta_l] \cdot \left(1 - G\left(\frac{\theta_l}{1-\kappa}\right)\right) \end{aligned}$$

Then

$$P_{01}[\theta_l' \leq \theta | \theta_C \cdot (1-\kappa) > \theta_l]$$

$$=P_{01}[\theta_l' \leq \theta | \theta_C \cdot (1-\kappa) > \theta_l, \theta_C \cdot (1-\kappa) > \theta_I] \cdot \overbrace{Pr[\theta_C \cdot (1-\kappa) > \theta_I | \theta_C \cdot (1-\kappa) > \theta_l]}^{\frac{1}{2}}$$

$$+ P_{01}[\theta_l' \leq \theta | \theta_C \cdot (1-\kappa) > \theta_l, \theta_C \cdot (1-\kappa) < \theta_I] \cdot \underbrace{Pr[\theta_C \cdot (1-\kappa) < \theta_I | \theta_C \cdot (1-\kappa) > \theta_l]}_{\frac{1}{2}}$$

$$= \frac{Pr\left[\frac{\theta_I}{1-\kappa} \leq \theta \middle| \theta_C \cdot (1-\kappa) > \theta_l, \theta_C \cdot (1-\kappa) > \theta_I\right] + Pr\left[\theta_C \cdot (1-\kappa) \leq \theta \middle| \theta_C \cdot (1-\kappa) > \theta_l, \theta_C \cdot (1-\kappa) < \theta_I\right]}{2}$$

$$= \frac{Pr\left[\theta_I \leq \theta \cdot (1-\kappa) \middle| \theta_C \cdot (1-\kappa) > \theta_l, \theta_C \cdot (1-\kappa) > \theta_I\right] + Pr\left[\theta_C \cdot (1-\kappa) \leq \theta \middle| \theta_C \cdot (1-\kappa) > \theta_l, \theta_C \cdot (1-\kappa) < \theta_I\right]}{2}$$

$$= \frac{Pr[\overbrace{\theta_I}^{Pareto(\theta_l,\alpha)} \leq \theta \cdot (1-\kappa) | \theta_C \cdot (1-\kappa) > \theta_l, \overbrace{\theta_C \cdot (1-\kappa)}^{Pareto(\theta_l,\alpha)} > \overbrace{\theta_I}^{Pareto(\theta_l,\alpha)}]}{2}$$

$$+ \frac{Pr[\overbrace{\theta_C \cdot (1-\kappa)}^{Pareto(\theta_l,\alpha)} \leq \theta | \theta_C \cdot (1-\kappa) > \theta_l, \overbrace{\theta_C \cdot (1-\kappa)}^{Pareto(\theta_l,\alpha)} < \overbrace{\theta_I}^{Pareto(\theta_l,\alpha)}]}{2}$$

$$= \frac{Pr[\overbrace{\min\{\theta_I, \theta_C \cdot (1-\kappa)\}}^{Pareto(\theta_l,2\alpha)} \leq \theta \cdot (1-\kappa) | \theta_C \cdot (1-\kappa) > \theta_l\}]}{2} + \frac{Pr[\overbrace{\min\{\theta_I, \theta_C \cdot (1-\kappa)\}}^{Pareto(\theta_l,2\alpha)} \leq \theta | \theta_C \cdot (1-\kappa) > \theta_l\}]}{2}$$

$$= \frac{G_{\theta_l,2\alpha}(\theta \cdot (1-\kappa)) + G_{\theta_l,2\alpha}(\theta)}{2}$$

This follows from the observation that if X_1, X_2 are i.i.d., then

$$Pr[\min\{X_1, X_2\} \leq t] = Pr[X_1 \leq t | X_1 \leq X_2]$$

Then the cdf of the reputation of the incumbent at $t = 2$ when the state targets only the challenger is given by

$$G_{\theta_l'|01}(\theta) = 1_{\{\theta \geq \theta_l\}} \cdot G\left(\frac{\theta_l}{1-\kappa}\right) + \left(\frac{G_{\theta_l,2\alpha}(\theta \cdot (1-\kappa)) + G_{\theta_l,2\alpha}(\theta)}{2}\right) \cdot \left(1 - G\left(\frac{\theta_l}{1-\kappa}\right)\right)$$

Next let's compute the distribution of θ_l' when the state only targets the incumbent at $t = 1$. Let's separate the analysis in two cases.

- Case $\theta_l \cdot (1-\kappa) > \theta_{\min}$:

$$
\begin{aligned}
P_{10}[\theta_l' \leq \theta] =& P_{10}[\theta_l' \leq \theta | \theta_C < \theta_l \cdot (1-\kappa)] \cdot P_{10}[\theta_C < \theta_l \cdot (1-\kappa)] \\
&+ P_{10}[\theta_l' \leq \theta | \theta_C > \theta_l \cdot (1-\kappa)] \cdot P_{10}[\theta_C > \theta_l \cdot (1-\kappa)] \\
=& 1_{\{\theta \geq \theta_l\}} \cdot G(\theta_l \cdot (1-\kappa)) + P_{10}[\theta_l' \leq \theta | \theta_C > \theta_l \cdot (1-\kappa)] \cdot (1 - G(\theta_l \cdot (1-\kappa)))
\end{aligned}
$$

Next

$$
\begin{aligned}
& P_{10}[\theta_l' \leq \theta | \theta_C > \theta_l \cdot (1-\kappa)] \\
=& P_{10}[\theta_l' \leq \theta | \theta_C > \theta_l \cdot (1-\kappa), \theta_C > \theta_I \cdot (1-\kappa)] \cdot Pr[\theta_C > \theta_I \cdot (1-\kappa) | \theta_C > \theta_l \cdot (1-\kappa)] \\
&+ P_{10}[\theta_l' \leq \theta | \theta_C > \theta_l \cdot (1-\kappa), \theta_C < \theta_I \cdot (1-\kappa)] \cdot Pr[\theta_C < \theta_I \cdot (1-\kappa) | \theta_C > \theta_l \cdot (1-\kappa)] \\
=& \frac{Pr[\theta_I \cdot (1-\kappa) \leq \theta | \theta_C > \theta_l \cdot (1-\kappa), \theta_C > \theta_I \cdot (1-\kappa)] + Pr[\frac{\theta_C}{1-\kappa} \leq \theta | \theta_C > \theta_l \cdot (1-\kappa), \theta_C < \theta_I \cdot (1-\kappa)]}{2} \\
=& \frac{Pr[\overbrace{\theta_I \cdot (1-\kappa)}^{Pareto(\theta_l \cdot (1-\kappa), \alpha)} \leq \theta | \theta_C > \theta_l \cdot (1-\kappa), \overbrace{\theta_C}^{Pareto(\theta_l \cdot (1-\kappa), \alpha)} > \overbrace{\theta_I \cdot (1-\kappa)}^{Pareto(\theta_l \cdot (1-\kappa), \alpha)}]}{2} \\
&+ \frac{Pr[\overbrace{\theta_C}^{Pareto(\theta_l \cdot (1-\kappa), \alpha)} \leq \theta \cdot (1-\kappa) | \theta_C > \theta_l \cdot (1-\kappa), \overbrace{\theta_C}^{Pareto(\theta_l \cdot (1-\kappa), \alpha)} < \overbrace{\theta_I \cdot (1-\kappa)}^{Pareto(\theta_l \cdot (1-\kappa), \alpha)}]}{2} \\
=& \frac{G_{\theta_l \cdot (1-\kappa), 2\alpha}(\theta) + G_{\theta_l \cdot (1-\kappa), 2\alpha}(\theta \cdot (1-\kappa))}{2}
\end{aligned}
$$

- Case $\theta_l \cdot (1-\kappa) < \theta_{\min}$:

$$
\begin{aligned}
P_{10}[\theta_l' \leq \theta] =& P_{10}[\theta_l' \leq \theta | \theta_I \cdot (1-\kappa) < \theta_{\min}] \cdot P_{10}[\theta_I \cdot (1-\kappa) < \theta_{\min}] \\
&+ P_{01}[\theta_l' \leq \theta | \theta_I \cdot (1-\kappa) > \theta_{\min}] \cdot P_{10}[\theta_I \cdot (1-\kappa) > \theta_{\min}] \\
=& 1_{\{\theta \geq \theta_{\min}\}} \cdot G_{\theta_l \cdot (1-\kappa), \alpha}(\theta_{\min}) + P_{10}[\theta_l' \leq \theta | \theta_I \cdot (1-\kappa) > \theta_{\min}] \cdot \left(1 - G_{\theta_l \cdot (1-\kappa), \alpha}(\theta_{\min})\right)
\end{aligned}
$$

Next

$$
\begin{aligned}
&P_{10}[\theta_l' \leq \theta | \theta_I \cdot (1-\kappa) > \theta_{\min}] \\
=&P_{10}[\theta_l' \leq \theta | \theta_I \cdot (1-\kappa) > \theta_{\min}, \theta_I \cdot (1-\kappa) < \theta_C] \cdot Pr[\theta_I \cdot (1-\kappa) < \theta_C | \theta_I \cdot (1-\kappa) > \theta_{\min}] \\
&+ P_{10}[\theta_l' \leq \theta | \theta_I \cdot (1-\kappa) > \theta_{\min}, \theta_I \cdot (1-\kappa) > \theta_C] \cdot Pr[\theta_I \cdot (1-\kappa) > \theta_C | \theta_I \cdot (1-\kappa) > \theta_{\min}] \\
=&\frac{P_{10}[\theta_I \cdot (1-\kappa) \leq \theta | \theta_I \cdot (1-\kappa) > \theta_{\min}, \theta_I \cdot (1-\kappa) < \theta_C] + P_{10}[\frac{\theta_C}{1-\kappa} \leq \theta | \theta_I \cdot (1-\kappa) > \theta_{\min}, \theta_I \cdot (1-\kappa) > \theta_C]}{2} \\
=&\frac{Pr[\overbrace{\theta_I \cdot (1-\kappa)}^{Pareto(\theta_{\min},\alpha)} \leq \theta | \theta_I \cdot (1-\kappa) > \theta_{\min}, \overbrace{\theta_I \cdot (1-\kappa)}^{Pareto(\theta_{\min},\alpha)} < \overbrace{\theta_C}^{Pareto(\theta_{\min},\alpha)}]}{2} \\
&+ \frac{Pr[\overbrace{\theta_C}^{Pareto(\theta_{\min},\alpha)} \leq \theta \cdot (1-\kappa) | \theta_I \cdot (1-\kappa) > \theta_{\min}, \overbrace{\theta_I \cdot (1-\kappa)}^{Pareto(\theta_{\min},\alpha)} > \overbrace{\theta_C}^{Pareto(\theta_{\min},\alpha)}]}{2} \\
=&\frac{G_{\theta_{\min},2\alpha}(\theta) + G_{\theta_{\min},2\alpha}(\theta \cdot (1-\kappa))}{2}
\end{aligned}
$$

Thus

$$
G_{\theta_l'|10}(\theta) = \begin{cases}
1_{\{\theta \geq \theta_l\}} \cdot G(\theta_l \cdot (1-\kappa)) \\
+ \frac{G_{\theta_l \cdot (1-\kappa),2\alpha}(\theta) + G_{\theta_l \cdot (1-\kappa),2\alpha}(\theta \cdot (1-\kappa))}{2} \cdot (1 - G(\theta_l \cdot (1-\kappa))) & , \theta_l \cdot (1-\kappa) > \theta_{\min} \\
1_{\{\theta \geq \theta_{\min}\}} \cdot G_{\theta_l \cdot (1-\kappa),\alpha}(\theta_{\min}) \\
+ \frac{G_{\theta_{\min},2\alpha}(\theta) + G_{\theta_{\min},2\alpha}(\theta \cdot (1-\kappa))}{2} \cdot \left(1 - G_{\theta_l \cdot (1-\kappa),\alpha}(\theta_{\min})\right) & , \theta_l \cdot (1-\kappa) < \theta_{\min}
\end{cases}
$$

Finally recall that

$$
G_{\theta_l'|11}(\theta) = G_{\theta_l'|00}(\theta) = 1_{\{\theta \geq \theta_l\}} G(\theta_l) + (1 - G(\theta_l)) \cdot G_{\theta_l, 2 \cdot \alpha}(\theta)
$$

□

Proof of lemma 4. Now let's compute $\mathbb{E}_{r_I^{t=1} r_C^{t=1}}\left[(\theta_l')^\eta\right] := \mathbb{E}\left[(\theta_l')^\eta \left| r_I^{t=1}, r_C^{t=1}\right.\right]$, where $\eta \in \{1-\alpha, \alpha\}$.

$$\mathbb{E}_{00}\left[(\theta_l')^\eta\right] = \mathbb{E}_{11}\left[(\theta_l')^\eta\right] = G(\theta_l)\cdot\theta_l^\eta + (1-G(\theta_l))\cdot\int_{\theta_l}^{\infty}\theta^\eta dG_{\theta_l,2\alpha}(\theta)$$
$$= G(\theta_l)\cdot\theta_l^\eta + (1-G(\theta_l))\cdot\frac{2\cdot\alpha}{2\cdot\alpha-\eta}\cdot\theta_l^\eta$$
$$= \left(G(\theta_l) + (1-G(\theta_l))\cdot\frac{2\cdot\alpha}{2\cdot\alpha-\eta}\right)\cdot\theta_l^\eta$$

$$\mathbb{E}_{01}\left[(\theta_l')^\eta\right] = G\left(\frac{\theta_l}{1-\kappa}\right)\cdot\theta_l^\eta + \left(1-G\left(\frac{\theta_l}{1-\kappa}\right)\right)\cdot\int_{\theta_l}^{\infty}\theta^\eta\frac{d}{d\theta}\left(\frac{G_{\theta_l,2\alpha}(\theta\cdot(1-\kappa)) + G_{\theta_l,2\cdot\alpha}(\theta)}{2}\right)$$
$$= G\left(\frac{\theta_l}{1-\kappa}\right)\cdot\theta_l^\eta$$
$$+ \left(1-G\left(\frac{\theta_l}{1-\kappa}\right)\right)\cdot\left(\frac{\overbrace{\int_{\frac{\theta_l}{1-\kappa}}^{\infty}\theta^\eta\cdot G'_{\theta_l,2\alpha}(\theta\cdot(1-\kappa))\cdot(1-\kappa)d\theta}^{(1-\kappa)^\eta\cdot\int_{\theta_l}^{\infty}\theta^\eta dG_{\theta_l,2\alpha}(\theta)} + \int_{\theta_l}^{\infty}\theta^\eta dG_{\theta_l,2\alpha}(\theta)}{2}\right)$$
$$= G\left(\frac{\theta_l}{1-\kappa}\right)\cdot\theta_l^\eta + \left(1-G\left(\frac{\theta_l}{1-\kappa}\right)\right)\cdot\left(\frac{(1-\kappa)^\eta\cdot\frac{2\cdot\alpha}{2\cdot\alpha-\eta}\cdot\theta_l^\eta + \frac{2\cdot\alpha}{2\cdot\alpha-\eta}\cdot\theta_l^\eta}{2}\right)$$
$$= \left(G\left(\frac{\theta_l}{1-\kappa}\right) + \left(1-G\left(\frac{\theta_l}{1-\kappa}\right)\right)\cdot\frac{2\cdot\alpha}{2\cdot\alpha-\eta}\cdot\frac{(1-\kappa)^\eta+1}{2}\right)\cdot\theta_l^\eta$$

To compute $\mathbb{E}_{10}\left[(\theta_l')^\eta\right]$ we will separate the analysis in two cases.

- Case $\theta_l\cdot(1-\kappa) > \theta_{\min}$:

$$\mathbb{E}_{10}\left[(\theta_l')^\eta\right] = G(\theta_l\cdot(1-\kappa))\cdot\theta_l^\eta$$
$$+ (1-G(\theta_l\cdot(1-\kappa)))\cdot\int_{\theta_l\cdot(1-\kappa)}^{\infty}\theta^\eta\frac{d}{d\theta}\left(\frac{G_{\theta_l\cdot(1-\kappa),2\cdot\alpha}(\theta) + G_{\theta_l\cdot(1-\kappa),2\alpha}(\theta\cdot(1-\kappa))}{2}\right)$$
$$= G(\theta_l\cdot(1-\kappa))\cdot\theta_l^\eta$$
$$+ (1-G(\theta_l\cdot(1-\kappa)))\cdot\frac{\int_{\theta_l\cdot(1-\kappa)}^{\infty}\theta^\eta dG_{\theta_l\cdot(1-\kappa),2\cdot\alpha}(\theta) + \overbrace{\int_{\theta_l}^{\infty}\theta^\eta G'_{\theta_l\cdot(1-\kappa),2\alpha}(\theta\cdot(1-\kappa))\cdot(1-\kappa)d\theta}^{(1-\kappa)^{-\eta}\cdot\int_{\theta_l\cdot(1-\kappa)}^{\infty}\theta^\eta dG_{\theta_l\cdot(1-\kappa),2\alpha}(\theta)}}{2}$$
$$= G(\theta_l\cdot(1-\kappa))\cdot\theta_l^\eta + (1-G(\theta_l\cdot(1-\kappa)))\cdot\frac{\frac{2\cdot\alpha}{2\cdot\alpha-\eta}\cdot(1-\kappa)^\eta\cdot\theta_l^\eta + \frac{2\cdot\alpha}{2\cdot\alpha-\eta}\cdot\theta_l^\eta}{2}$$
$$= \left(G(\theta_l\cdot(1-\kappa)) + (1-G(\theta_l\cdot(1-\kappa)))\cdot\frac{2\cdot\alpha}{2\cdot\alpha-\eta}\cdot\frac{(1-\kappa)^\eta+1}{2}\right)\cdot\theta_l^\eta$$

- Case $\theta_l \cdot (1-\kappa) < \theta_{\min}$:

$$
\begin{aligned}
\mathbb{E}_{10}\left[(\theta_l')^\eta\right] =& G_{\theta_l\cdot(1-\kappa),\alpha}\left(\theta_{\min}\right)\cdot\theta_{\min}^\eta \\
&+\left(1-G_{\theta_l\cdot(1-\kappa),\alpha}\left(\theta_{\min}\right)\right)\cdot\frac{\int_{\theta_{\min}}^\infty \theta^\eta dG_{\theta_{\min},2\alpha}(\theta)+\int_{\frac{\theta_{\min}}{1-\kappa}}^\infty \theta^\eta dG_{\theta_{\min},2\alpha}\left(\theta\cdot(1-\kappa)\right)}{2} \\
=& G_{\theta_l\cdot(1-\kappa),\alpha}\left(\theta_{\min}\right)\cdot\theta_{\min}^\eta \\
&+\left(1-G_{\theta_l\cdot(1-\kappa),\alpha}\left(\theta_{\min}\right)\right)\cdot\frac{\int_{\theta_{\min}}^\infty \theta^\eta dG_{\theta_{\min},2\alpha}(\theta)+\overbrace{\int_{\frac{\theta_{\min}}{1-\kappa}}^\infty \theta^\eta G'_{\theta_{\min},2\alpha}\left(\theta\cdot(1-\kappa)\right)\cdot(1-\kappa)d\theta}^{(1-\kappa)^{-\eta}\cdot\int_{\theta_{\min}}^\infty \theta^\eta dG_{\theta_{\min},2\alpha}(\theta)}}{2} \\
=& G_{\theta_l\cdot(1-\kappa),\alpha}\left(\theta_{\min}\right)\cdot\theta_{\min}^\eta \\
&+\left(1-G_{\theta_l\cdot(1-\kappa),\alpha}\left(\theta_{\min}\right)\right)\cdot\frac{\frac{2\cdot\alpha}{2\cdot\alpha-\eta}\cdot\theta_{\min}^\eta+(1-\kappa)^{-\eta}\cdot\frac{2\cdot\alpha}{2\cdot\alpha-\eta}\cdot\theta_{\min}^\eta}{2} \\
=&\left(G_{\theta_l\cdot(1-\kappa),\alpha}\left(\theta_{\min}\right)+\left(1-G_{\theta_l\cdot(1-\kappa),\alpha}\left(\theta_{\min}\right)\right)\cdot\frac{2\cdot\alpha}{2\cdot\alpha-\eta}\cdot\frac{(1-\kappa)^{-\eta}+1}{2}\right)\cdot\theta_{\min}^\eta
\end{aligned}
$$

□

Proof of 4. Sketch of proof incomplete.

First assume $\alpha \in \left(\frac{1}{2}, 1\right)$

$$
\mathbb{E}_{r_I r_C}[U_S^*|\theta_l] = -\begin{cases}
F\left[\theta_l,\theta_{\min}\right]+(1-\kappa)\cdot\mathbb{E}_{00}\left[F[\theta_{\min},\theta_l']|\theta_l]\right] & ,\ (r_I,r_C)=(0,0)\\
F\left[\theta_l,(1-\kappa)\cdot\theta_{\min}\right]+(1-\kappa)\cdot\mathbb{E}_{01}\left[F[\theta_{\min},\theta_l']|\theta_l]\right] & ,\ (r_I,r_C)=(0,1)\\
F\left[(1-\kappa)\cdot\theta_l,\theta_{\min}\right]+(1-\kappa)\cdot\mathbb{E}_{10}\left[F[\theta_{\min},\theta_l']|\theta_l]\right] & ,\ (r_I,r_C)=(1,0)\\
(1-\kappa)\cdot F\left[\theta_l,\theta_{\min}\right]+(1-\kappa)\cdot\mathbb{E}_{11}\left[F[\theta_{\min},\theta_l']|\theta_l]\right] & ,\ (r_I,r_C)=(1,1)
\end{cases}
$$

Where

$$\mathbb{E}_{r_I r_C}\left[F[\theta_{\min},\theta_l']|\theta_l]\right] = (1-\kappa)\cdot\frac{\alpha}{2\alpha-1}\cdot\theta_{\min}^{\alpha}\cdot$$

$$\begin{cases} \left(G(\theta_l)+(1-G(\theta_l))\cdot\frac{2\cdot\alpha}{3\cdot\alpha-1}\right)\cdot\theta_l^{1-\alpha} & ,\ (r_I,r_C)\in\{(0,0),(1,1)\} \\ \left(G\left(\frac{\theta_l}{1-\kappa}\right)+\left(1-G\left(\frac{\theta_l}{1-\kappa}\right)\right)\cdot\frac{2\cdot\alpha}{3\cdot\alpha-1}\cdot\frac{(1-\kappa)^{1-\alpha}+1}{2}\right)\cdot\theta_l^{1-\alpha} & ,\ (r_I,r_C)=(0,1) \\ \left(G(\theta_l\cdot(1-\kappa))+(1-G(\theta_l\cdot(1-\kappa)))\cdot\frac{2\cdot\alpha}{3\cdot\alpha-1}\cdot\frac{(1-\kappa)^{1-\alpha}+1}{2}\right)\cdot\theta_l^{1-\alpha} & ,\ (r_I,r_C)=(1,0) \text{ and } \theta_l\cdot(1-\kappa)>\theta_{\min} \\ \left(G_{\theta_l\cdot(1-\kappa),\alpha}(\theta_{\min})+\left(1-G_{\theta_l\cdot(1-\kappa),\alpha}(\theta_{\min})\right)\cdot\frac{2\cdot\alpha}{3\cdot\alpha-1}\cdot\frac{(1-\kappa)^{-(1-\alpha)}+1}{2}\right)\cdot\theta_{\min}^{1-\alpha} & ,\ (r_I,r_C)=(1,0) \text{ and } \theta_l\cdot(1-\kappa)<\theta_{\min} \end{cases}$$

□