Fairness and well-being measurement*

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March 2014; preliminary and incomplete

Abstract

We assume that economic justice requires resources to be allocated fairly. We construct individual well-being measures. These measures are required to respect agents' preferences. Interpersonal comparisons are required to depend on comparisons of the bundles of resources consumed by agents. We show that it is essential to take the nature of the goods consumed by agents into account. When all goods are perfectly divisible and more of the goods is always preferred to less, we axiomatically justify two main families of well-being measures. They generalize the concepts of money-metric utility and ray utility. When only one good has those properties, then other requirements need to be imposed. We also justify two main families of well-being measures. In the general case of goods of all nature, the two pairs of families can be combined.

JEL Classification: D63, I32.

Keywords: fairness, well-being measure, preferences.

*The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement n° 269831.

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1 Introduction

Notions of individual well-being are important both in political philosophy and in welfare economics. Whether one defines the just society or one evaluates social policies, defining well-being is typically needed. The questions of how to measure well-being and how to aggregate well-being levels are typically raised at the same time, though, and general solutions to these questions may come from arguments that mix them.

There are, of course, many competing ways of defining individual well-being. One main theory of well-being used by economists, called the theory of fair allocation, consists of comparing agents' well-being on the ground of the bundles of resources that they consume. This is in line with Rawls and Dworkin's view that economic justice should be defined in terms of equality of resources. There are cases in which this is easily done. If all agents are assumed to have the same preferences, as it is the case in the optimal taxation literature following Mirrlees' (1971) seminal contribution, then the well-being measure is simple required to be consistent with these common preferences. If all agents have (possibly heterogenous) quasi-linear preferences in money, then the money measure of satisfaction level is natural and creates an easy way of comparing well-being. Identical or quasi-linear preferences are extremely common assumptions.

It is not always relevant to make those assumptions, however. If some agents are close to their liquidity constraints, for instance, it is hard to assume away all income effects. One may also wish to take account of agents' different ways of reacting to policies. As soon as one acknowledge that there are income effects and that agents have heterogenous preferences, it is no longer clear how well-being should be measured.

Many authors have directly or indirectly raised that question. The question has been raised directly in the literature on consumer surplus and how it can be measured from changes in prices and consumed quantities. This abundant literature has culminated in Samuelson's (1974) and Samuelson and Swamy's (1974) concept of money-metric utility, and Samuelson's (1977) and Pazner's (1979) concept of ray utility, that will play a crucial rle in what follows. The money-metric utility consists in a priori fixing a vector of prices and measuring well-being by the budget, at those prices, that leaves the agent indifferent with her actual consumption. The ray utility consists in a priori fixing a ray of goods in the consumption set of the agents and measuring well-being by the only bundle of resources along that ray that leaves an agent indifferent with her actual consumption.

The question of measuring well-being has also been raised indirectly in the literature on fair allocation. The objective of that literature, recently surveyed by Thomson (2011) and Fleurbaey and Maniquet (2011), is to define fair ways of allocating resources. Solutions from that literature can be seen as answering simultaneously the questions of how to measure individual well-being and how to aggregate them.

In this paper, we revisit certain recent results obtained in the fair allocation literature by addressing the question of how to measure individual well-being without referring a priori to prices and without mixing the issue of well-being measurement with that of well-being aggregation. Starting with an abstract model of individual well-being, we axiomatically study how to construct interpersonal well-being comparisons based on comparisons of the bundles of resources they consume and their preferences. This is in line with undertakings recently launched by Fleurbaey and Tadenuma (2013) and Fleurbaey and Blanchet (2013).

The axioms we study force us to distinguish between two sub models, depending on the nature of the goods. When goods are infinitely divisible, their measure is cardinal, and more of any good is always preferred to less, two families of measures emerge. One family is consistent with the idea that comparing well-being requires to determine worst preferences. Worst preferences are preferences that make the experience of consuming any bundles of resources worse than with any other preferences. Worst preferences are naturally connected to the difficulty of trading off between goods. The other family is consistent with the idea that comparing well-being requires to determine best preferences. Best preferences are the ones that make the experience of consuming any bundles of resources better than with any other preferences. Best preferences are naturally connected to the ability to trade-off between goods.

This first set of results sheds some light to the previous literature on well-being measures. Indeed, the ray utility belongs to the first family of well-being measures we obtain. Our results give an axiomatic characterization of that measure, but it also shows that many other measures can be similarly justified. Note that this theory has never found deep justification in terms of selecting among the possible rays one that deserves special attention. Our result do not make any progress on that question.

Money-metric utility belongs to the second family of well-being measures we obtain. Again, our results can be viewed as providing an axiomatic justification to that measure, but they also show that other measures can receive similar justification. The literature on equivalent income has never solved the question of which price vector to select, and our study does not progress on that issue either.

Our results also shed some light on the theory of fair allocation. In that theory, two prominent allocation rules receive considerable justification. An allocation rule identifies the set of best allocations among the feasible ones. The first one is the egalitarian equivalent allocation rule, introduced by Pazner and Schmeidler (1978) and later characterized, among others, by Moulin (1987) and Sprumont and Zhou (1999). It consists in allocating goods in such a way that each agent is indifferent between the bundle she is assigned and a common, reference bundle. This is consistent with a way of measuring well-being that belongs to our first family of measures. Of course, the egalitarian equivalent allocation rule also shows how well-being should be aggregated: all agents should have the same well-being. On the other hand, the allocation rule is silent about how to allocate resources when equality is not feasible.

The second main allocation rule is the equal income Walrasian rule, first studied by Kolm (1968, 1972) and Varian (1974). It consists in allocating goods in such a way that the resulting allocation can be thought of a competitive equilibrium allocation from an equal split of the resources. This allocation rule can be decomposed into a way of defining well-being and a way of aggregating it. The way of defining well-being is by looking at equivalence with Walrasian budgets computed at equilibrium prices, those that would prevail if resources were first allocated equally among all agents. This well-being measure belongs to our second family of measures. That theory can be viewed as solving the question of which prive vector to use, but is silent about how to allocate resources when the equal income Walrasian allocations are not feasible.

The theory of fair allocation has recently looked as social ordering functions instead of allocation rules. A social ordering function is a complete ordering on allocations. The study of social ordering functions, surveyed in Fleurbaey and Maniquet (2011), has provided us with two main conclusions. The first conclusion is that there was one and only one prominent aggregator of individual well-being levels, the maximin aggregator. That is, simple and weak requirements on social ordering functions force us to maximize the lowest well-being level among agents. The second conclusion is that many different individual well-being measures receive justification from fairness requirements. Many of those measures are of the equivalence type: the well-being of an agent is measured with respect to the bundle of goods, in a set of reference bundles, that leaves this agent indifferent with her actual consumption. Some other measures are closer to the money-metric type: the well-being of an agent is measured with respect to the income that leaves her indifferent to her actual bundle, with prices being chosen so as to maximize the minimal income (see Fleurbaey and Maniquet, 2008 and 2011).

The results in this paper can be viewed as complementary to the study of social ordering functions. Indeed, social ordering function may be seen as trying to accomplish too much. Our starting point in this paper is the distinction between the effort of defining well-being and the effort of aggregating well-being across agents. This is particularly important if one considers that maximin is too extreme. Then, indeed, our results open new possibilities. We show that individual well-being measures can be defined independently to the question of how to aggregate them. As a consequence, there is no reason to limit the aggregator to the maximin one. Our notions of well-being measures perfectly fit the framework of Bergson-Samuelson social welfare functions, and all well-being aggregator analyzed in the literature on social welfare functionals (surveyed, among others, in Bossert and Weymark, 2004, and d'Aspremont and Gevers, 2002), for instance, can be applied to well-being indices as they are defined in this paper.

Our results come close to and are inspired by the recent study of Decancq, Fleurbaey and Maniquet (2014) on poverty measures. A measure of individual poverty is no more than the inverse of a well-being measure. The authors of that paper axiomatized a poverty measure that consists in first defining an individual poverty measure consistent with the ray utility function and then aggregating individual poverty in a way that is only required to be consistent with dominance. Some of their results are reproduced here with the only difference that the object we are interested in here is a well-being measure.

Then, we proceed by making different assumptions on the nature of the goods. We allow all goods but one to fail to be divisible or cardinal, and we drop the assumption that more of them is always preferred. All those goods are named attributes. Under that alternative assumption, the axioms we are interested in do not deliver the same messages as in the original model. Consequently, we define alternative axioms, appropriate for the alternative nature of the goods. Our second set of results gives us axiomatic characterizations of two families of well-being measures. They both belong to the well-known family of equivalent income measures. This family is mostly meaningful if one interprets the divisible and cardinal dimension as incomes, which our first set of results prevent us from doing. Equivalent income is the income that would leave an agent indifferent between her actual situation and receiving that income and consuming some specified attribute. The two families of well-being measures we obtain correspond to two different ways of specifying the reference attribute. They also correspond to the measures that were recently proposed and axiomatized by Fleurbaey and Blanchet (2013).

We end up with two pairs of families of well-being measures, each pair relevant under specific assumptions on the nature of the goods. In a more general model with goods of all possible nature, each combination of the different families is possible. We obtain, therefore, four families of well-being measures, even if two of them can be seen as more consistent. This teaches us that the so-called equivalent incomes can be replaced with other measures in which equivalence is not taken with respect to incomes but with respect to either of the well-being measures that are prominent in the case of divisible and cardinal goods.

A last lesson can be drawn from this inquiry. Whether or not goods are marketed, transferable, and private does not matter. What matters is whether or not they are cardinal and whether more is always preferred to less. This follows from our restriction to individual consumption by an agent, and our decision not to raise the question of the aggregation of well-being.

The well-being measures that we justify in this paper are consistent with the view that economic justice arises from a fair allocation of resources. The measures we propose are solutions to the difficulty arising from the heterogeneity of preferences. We have not addressed, however, the difficulty arising from heterogeneity in needs or in abilities.

2 A model of well-being measurement

We begin by assuming that there are K divisible goods, and quantities of goods are cardinally measurable (so that, for instance, arithmetic averages of quantities are meaningful). The consumption set is $\mathbb{R}^K_+$. We are interested in measuring well-being when agents consume bundles in a set $X \subseteq \mathbb{R}^K_+$. Agents have continuous, convex and monotone[1] preferences over $\mathbb{R}^K_+$. We let $\mathcal{R}$ denote the set of all such preferences. A well-being measure over X

[1]We use $>, \geqslant$ and $\gg$ to denote the vector inequalities. Preferences R are monotone if and only if $x > x'$ implies $x \, R \, x'$ and $x \gg x'$ implies $x \, P \, x'$.

is a function $W : X \times \mathcal{R} \to \mathbb{R}$, such that $W(x, R)$ is the well-being level of an agent consuming bundle x and having preferences R. Note that we require from W that it gives us a well-being function for all preferences. This corresponds to the typical universal domain requirement.

Throughout the paper, we require the following two conditions on W. First, W is continuous in x. Second, W respects the preferences, in the sense that for all $x, x' \in X$, $R \in \mathcal{R}$,

$$x\, R\, x' \Rightarrow W(x, R) \geq W(x', R) \text{ and } x\, P\, x' \Rightarrow W(x, R) > W(x', R).$$

The latter condition is reminiscent of Pareto efficiency in the social choice literature. Here, it represents our desire to define well-being in a way that is consistent with what agents themselves think about how the different dimensions of life should be aggregated.

The following terminology will prove useful. For $x \in \mathbb{R}^K_+$, $R \in \mathcal{R}$, $L(x, R), U(x, R)$ and $I(x, R)$ denote the lower, upper and indifference contour of R at x, respectively, that is,

$$\begin{aligned} L(x, R) &= \{x' \in \mathbb{R}^K_+ | x\, R\, x'\}, \\ U(x, R) &= \{x' \in \mathbb{R}^K_+ | x'\, R\, x\}, \\ I(x, R) &= L(x, R) \cap U(x, R). \end{aligned}$$

The first axiom we define captures the idea that well-being comparisons should be made on the basis of the resources that agents consume. We find it a weak axiom and we will impose it throughout the paper. It requires that the well-being of an agent at some bundle be declared larger than that of another agent at another bundle when the following two conditions are satisfied. First, both agents prefer the former bundle to the latter bundle. Second, both agents prefer any bundle that the former agent finds indifferent to the former bundle over any bundle that the latter agent finds indifferent to the latter bundle. These two conditions amount to say that the upper contour set at the former bundle does not intersect the lower contour set at the latter bundle.

Axiom 1 Nested Contour
For all $x, x' \in X$, $R, R' \in \mathcal{R}$, if $U(x, R) \cap L(x', R') = \emptyset$, then $W(x, R) > W(x', R')$.

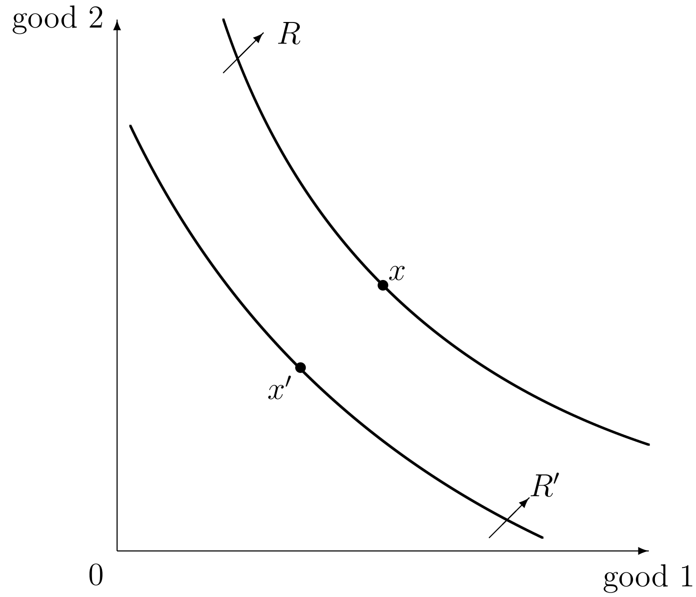

Figure 1: *Nester Contour*: $W(x,R) > W(x',R')$.

A similar axiom was introduced in Decancq, Fleurbaey and Maniquet (2013) in their study of poverty measurement (individual poverty measures are opposite functions to well-being measures).

We may also note a relationship between this axiom and the celebrated no-envy axiom of the literature on fair allocation. An allocation is a no-envy allocation if no agent strictly prefers the bundle assigned to another agent to her own. It is known that a no-envy allocation may be Pareto indifferent to another allocation in which one agent envies another agent. In terms of well-being measurement, that means that we cannot base our well-being comparisons on envy considerations. A well-being measure that would require that an agent envying another agent's consumption should have a lower well-being is impossible to define. Let us think, indeed, at a pair of bundles $x, x' \in X$ and a pair of preferences $R, R' \in \mathcal{R}$ with crossing indifference surfaces such that $x' \, P \, x$ and $x \, P' \, x'$. It would be impossible to give values to $W(x,R)$ and $W(x',R')$, as the no-envy requirement would read $W(x,R) < W(x',R')$ and $W(x',R') < W(x,R)$.

The above axiom offers a way to reconcile no-envy and the requirement that a well-being measure be respectful to individual preferences. In the comparison considered in the axiom, agent R', who consumes x', envies agent R, who consumes x, but, moreover, agent R' consuming any bundle she deems equivalent to x' would envies agent R consuming any bundle she deems equivalent to x.

Nested Contour is equivalent to the following axiom. It requires that the well-being of an agent consuming a given bundle only depend on her indifference surface at that bundle.

Axiom 2 UNCHANGED INDIFFERENCE SURFACE INDEPENDENCE
For all $x \in X$, $R, R' \in \mathcal{R}$, *if* $I(x, R) = I(x, R')$, *then* $W(x, R) = W(x, R')$.

Lemma 1 *A well-being measure* W *satisfies* Nested Contour *if and only if it satisfies* Unchanged Indifference Surface Independence*.*

Unchanged Indifference Surface Independence is also reminiscent to a series of axioms in the theory of fair allocation and social ordering functions that require independence of the selection of the best allocation to changes in preferences that do not affect the indifference surface through the selection.

We consider that a well-being measure violating *Nested Contour* would be hard to justify, if one wants to be consistent with the idea that economic justice comes from equality of resources. Our strategy from now on will be to propose strengthening of *Nested Contour* and to study their implications. We will propose three ways of strengthening the axiom, which will turn out to be incompatible with each other, and that will lead us to defining three families of well-being measures.

3 Lower Contour Inclusion

Nested Union may be redefined in the following way. If the lower contour set of one agent at her consumption lies in the interior of the lower contour set of another agent at her consumption, then the well-being of the former agent is strictly lower than that of the latter agent. This idea can be immediately extended to include several agents in the following way. If the lower contour set of one agent lies in the interior of the union of lower contour sets of other agents, then the well-being of the former agent is strictly lower than that of at least one of the latter agents.

Axiom 3 Lower Contour Inclusion
For all $x, x', x'' \in X$, $R, R', R'' \in \mathcal{R}$, *if* $(L(x'', R'') \cap X) \subset interior[(L(x, R) \cup L(x', R')) \cap X]$, *then* $W(x'', R'') < \max\{W(x, R), W(x', R')\}$.

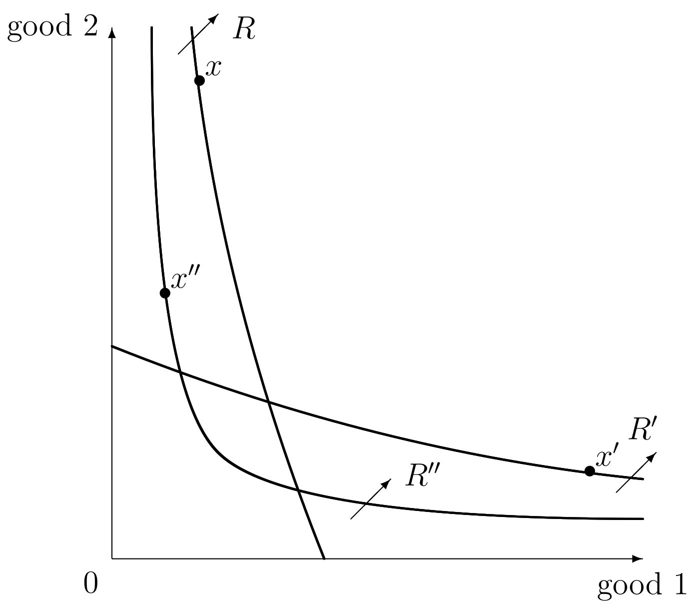

Figure 2: *Lower Contour Inclusion*: $W(x'', R'') < \max\{W(x, R), W(x', R')\}$.

This axiom is clearly more controversial than *Nested Contour*. The situation is that the bundle x'' consumed by agent R'' is either considered worse than x by agent R, or worse than x' by agent R', or both. This is why it would be hard to justify that agent R'' is strictly better-off than the two other agents.

Our first result provides us with a characterization of the well-being measures that satisfy *Lower Contour Inclusion*. This characterization is in terms of the existence of worst preferences. We can say that preferences $R^w \in \mathcal{R}$ are the worst preferences if the well-being of an agent with those preferences is always lower than that of any other agent whatever the (common) bundle they consume.

Axiom 4 Worst Preferences
There exists $R^w \in \mathcal{R}$ *such that for all* $x \in X$, $R \in \mathcal{R}$, $W(x, R^w) \leq W(x, R)$.

Lower Contour Inclusion turns out to be equivalent to *Nested Contour* and *Worst Preferences*. Moreover, once worst preferences R^w are chosen and the well-being measure of the worst preferences $W(\cdot, R^w)$ is determined, we have a unique and well-defined well-being measure.

Theorem 1 *Let X be a convex and compact set. A well-being measure W over X satisfies* Lower Contour Inclusion *if and only if it satisfies* Nested Contour *and* Worst Preferences*. Moreover, for worst preferences $R^w \in \mathcal{R}$, the well-being measure is defined by: for all $x \in X$ and $R \in \mathcal{R}$:*

$$W(x, R) = \max_{x' \in L(x,R)} W(x', R^w).$$

Note that the existence of a maximal element for R^w in $L(x, R)$ is guaranteed by compactness of X. *Worst Preferences* constructs comparability between any preferences and the worst preferences. The theorem proves that when it is combined with *Nested Contour*, it constructs comparability between any pair of preferences.

At this stage, any preference relation $R \in \mathcal{R}$ can be chosen to be the worst one and the axioms are satisfied. Some preferences, though, are more natural candidates to be worst preferences than others. We think that Leontieff preferences are natural candidates. Preferences R^ℓ are Leontieff if there exist K parameters $\ell_k \in \mathbb{R}$, $k \in K$, such that

$$x \, R^\ell \, x' \Leftrightarrow \min_{k \in K} \frac{x_k}{\ell_k} \geq \min_{k \in K} \frac{x'_k}{\ell_k}.$$

An agent with Leontieff preferences is unable to substitute one good for another. When such an agent consumes bundle x, her well-being is entirely determined by $\min_{k \in K} \frac{x_k}{\ell_k}$, that is, the good in which this agent feels most deprived.

If a well-being measure satisfies *Worst Preferences* with $R^w = R^\ell$ for some $\ell \in \mathbb{R}^K$, then the well-being of an agent is measured by the bundle that is proportional to ℓ to which this agent is indifferent. To put it differently, we say that $W(x, R) = W(x', R')$ if and only if there exists some number $\lambda \in \mathbb{R}_+$ such that $x \, I \, \lambda\ell$ and $x' \, I' \, \lambda\ell$ as well. All the well-being measures satisfying this property are ordinally equivalent to W^ℓ, defined by: for all $x \in X$, all $R \in \mathcal{R}$,

$$W^\ell(x, R) = w \Leftrightarrow x \, I \, w\ell.$$

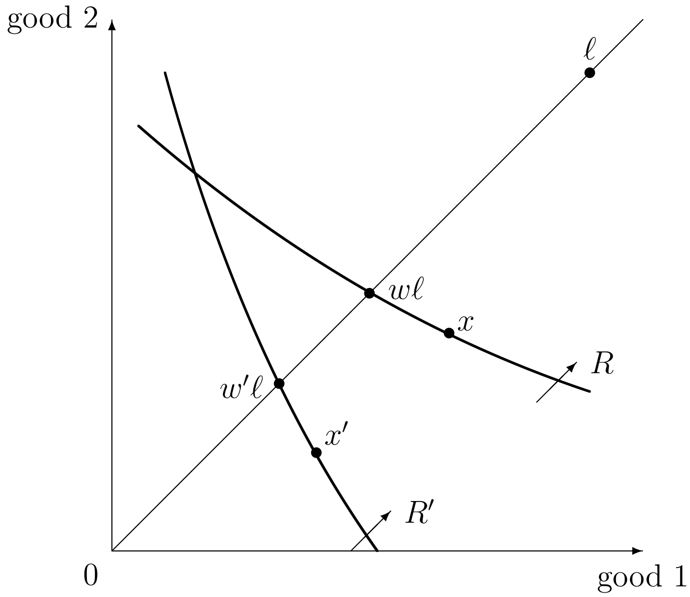

Figure 3: Illustration of W^ℓ: $W^\ell(x, R) = w, W^\ell(x', R') = w'$.

Observe that the set of bundles $r \in \mathbb{R}^K_+$ such that $r = \lambda\ell$ for some $\lambda \in \mathbb{R}_+$ is a ray in $\mathbb{R}^K_+$. This way of measuring well-being was suggested by Samuelson (1977), Deaton (1979) and Pazner (1979), but none of them justified it axiomatically. If it is combined with an egalitarian aggregator, it comes close to allocation rules and social ordering functions axiomatized in the literature on fair allocation.

Well-being measures consistent with $R^w = R^\ell$ for some $\ell \in \mathbb{R}^K$ do satisfy the axioms of the theorem above, but many others do. A natural question is then whether natural axioms stronger than *Lower Contour Inclusion* can be introduced and whether they would be satisfied by taking Leontieff as worst preferences. The next axiom partly answers that question.

We need the following terminology. Let $x \in X$ and $R, R', R'' \in \mathcal{R}$. We say that R'' is intermediary to R and R' at x, formally, $R'' \in intermediary(R, R', x)$, if the indifference surface of R'' at x is everywhere between that of R and that of R', that is, if for all $x' \in X$ such that $x' \, I'' \, x$, either $x \, P \, x'$ and $x' \, P' \, x$, or $x' \, P \, x$ and $x \, P' \, x'$, or both $x \, I \, x'$ and $x' \, I' \, x$. The following axiom requires that if some preferences are intermediary to two other preferences at a bundle, then the well-being at this bundle associated to the former preferences should also be intermediary to the well-being associated to the latter preferences.

Axiom 5 INTERMEDIARY PREFERENCES
For all $x \in X$, $R, R', R'' \in \mathcal{R}$, if $R'' \in intermediary(R, R', x)$, then $W(x, R'') \in [W(x, R), W(x, R')]$.

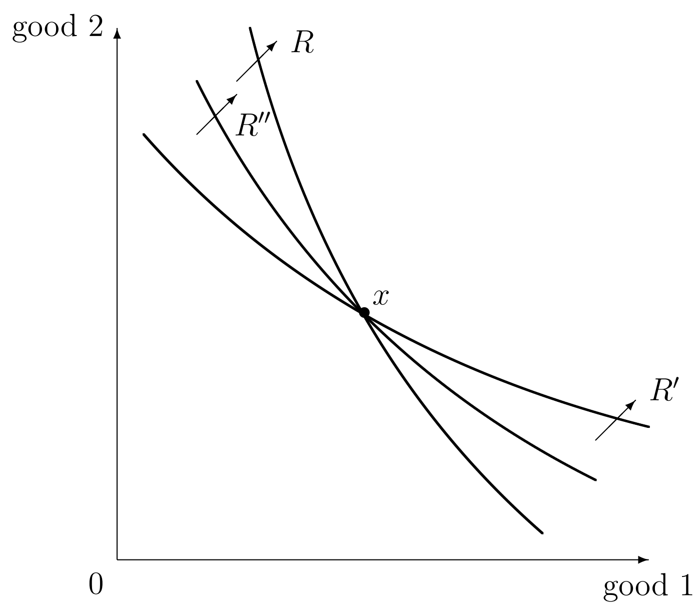

Figure 4: *Intermediary Preferences*: $W(x, R'') \in [W(x, R), W(x, R')]$.

This axiom may look natural: If some preference relation is intermediary between two other preference relations, it is natural that the well-being measure be also intermediary. The critical point, of course, is that there are many different ways of defining what intermediary preferences are. Here, we adopt a definition that is based on the indifference surfaces of the different preference relations at one bundle. We will see later on in the paper that we can define intermediary preferences differently.

When *Nested Contour* is imposed, *Intermediary Preferences* is stronger than *Lower Contour Inclusion*.

Lemma 2 *If a well-being measure W satisfies* Nested Contour *and* Intermediary Preferences*, it satisfies* Lower Contour Inclusion*.*

The next result characterizes the family of well-being measures that satisfy *Nested Contour* and *Intermediary Preferences*. The key notion is that of a monotone consumption path. It is a set of bundles, starting at the origin of the consumption set and increasing continuously and unboundedly towards strictly larger bundles. We say that $P \subset \mathbb{R}^K_+$ is a monotone consumption path if

- $0^K = (0, \ldots, 0) \in P$,
- for all $x, x' \in P$, either $x \ll x'$, or $x' \ll x$ or $x = x'$,
- P is homeomorphic to $\mathbb{R}_+$,
- for all $r \in \mathbb{R}^K_+$, there exists $x \in P$ such that $x \ll r$.

Rays are special exemples of monotone consumption paths. Observe that a monotone consumption path P is constructed in such a way that for all $x \in \mathbb{R}^K_+$, all $R \in \mathcal{R}$, there exists one and only one $p \in P$ such that $x \, I \, p$. The next theorem proves that any well-being measure satisfying *Nested Contour* and *Intermediary Preferences* has the following shape: Well-being is measured by the intersection between the indifference surface of an agent at a bundle and a monotone consumption path.

Theorem 2 *If a well-being measure satisfies* Nested Contour *and* Intermediary Preferences*, then there exists a monotone consumption path P and a strictly increasing function $w : P \to \mathbb{R}_+$ such that for all $x \in \mathbb{R}^K_+$, all $R \in \mathcal{R}$, $W(x, R) = w(p)$ for $p \in P$ such that $x \, I \, p$.*

Well-being measures grounded on the axiom of *Worst Preferences* with Leontieff as worst preferences belong to the family of measures characterized by the above axiom. They are not the only ones, though. Monotone consumption paths need not be straight rays.

It is interesting to study the difference between the above theorem, which does not give any special status to rays among the monotone consumption paths, and other results that do. In Decancq, Fleurbaey and Maniquet (2013), for instance, poverty levels are measured along rays, and that comes from a requirement that transfers between poor agents should decrease aggregate poverty. It is well known that this kind of transfer principle is extremely difficult to satisfy in multidimensional consumption sets. As a consequence, they impose transfers to be progressive only in convex sets of bundles, and these convex sets turn out to have to be rays. In the literature on fair allocation rules or social ordering functions, egalitarian equivalence along rays of bundles is obtained by imposing axioms that refer to equal split of the available resources. In both cases, rays are deduced from axioms that deal with both the measurement of well-being/poverty and the aggregation of such measures. As a result, our inquiry shows that if we want to disentangle the question of well-being comparison from the aggregation question, more possibilities emerge for the definition of well-being measures.

4 Convex Hull Inclusion

In this section, we study another strengthening of *Nested Contour*. It also refers to the idea that some preferences are intermediary. Consider $x, x', x'' \in X$ and $R, R', R'' \in \mathcal{R}$. One obvious way in which we could say that x'' is intermediary between x and x' is if x'' lies between the two other bundles, that is $x'' = \alpha x + (1-\alpha)x'$, for some $\alpha \in [0,1]$. Of course, this is not sufficient to conclude that we should have $W(x, R'') \in [W(x,R), W(x,R')]$ because both x and x' might be good for R and R' respectively, in some sense of "goodness", whereas x'' is not that good for R''. This can be avoided if we furthermore require that all bundles that are indifferent to x'' for R'' be also intermediary between two bundles that are indifferent to x for R and x' for R'. More formally, (x'', R'') is intermediary between (x, R) and (x', R') if for all $y'' \in X$ such that $y'' \, I'' \, x''$, there exist $y, y' \in X$ such that $y \, I \, x$ and $y' \, I' \, x'$ and $y'' = \alpha y + (1-\alpha)y'$, for some $\alpha \in [0,1]$. Observe that this is equivalent to requiring that $U(x'', R'')$ be included in the convex hull of the union of $U(x, R)$ and $U(x', R')$.[2] We can then state the axiom that if (x'', R'') is intermediary between (x, R) and (x', R') in this sense, then the well-being at (x'', R'') cannot be lower than that of both (x, R) and (x', R'). Let CH denote the convex hull operator.

Axiom 6 Convex Hull Inclusion
For all $x, x', x'' \in X$, all $R, R', R'' \in \mathcal{R}$, if

$$(U(x'', R'') \cap X) \subset interior[(CH(U(x,R) \cap U(x', R'))) \cap X],$$

then $W(x'', R'') > \min\{W(x,R), W(x', R')\}$.

[2]The convex hull of a set is the smallest convex set containing that set.

Observe that *Convex Hull Inclusion* is logically stronger than *Nested Contour* (simply take $(x, R) = (x', R')$).

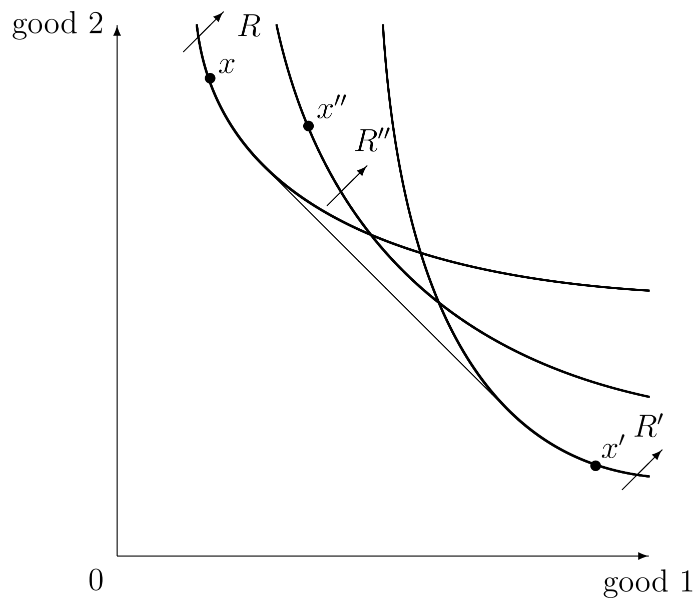

Figure 5: *Convex Hull Inclusion*: $W(x'', R'') > \min\{W(x, R), W(x', R')\}$.

Our next result provides us with a characterization of the well-being measures that satisfy *Convex Hull Inclusion*. This characterization is in terms of the existence of best preferences. We can say that preferences $R^b \in \mathcal{R}$ are the best preferences if the well-being of an agent with those preferences is always above that of any other agent whatever the (common) bundle they consume.

Axiom 7 Best Preferences
There exists $R^b \in \mathcal{R}$ such that for all $x \in X$, $R \in \mathcal{R}$, $W(x, R^w) \geq W(x, R)$.

Convex Hull Inclusion turns out to be equivalent to *Nested Contour* and *Best Preferences*. Moreover, once best preferences R^b are chosen and the well-being measure of the best preferences $W(\cdot, R^b)$ is determined, we have a unique and well-defined well-being measure.

Theorem 3 *Let X be a convex and compact set. A well-being measure W over X satisfies* Convex Hull Inclusion *if and only if it satisfies* Nested Contour *and* Best Preferences. *Moreover, for best preferences $R^b \in \mathcal{R}$, the well-being measure is defined by: for all $x \in X$ and $R \in \mathcal{R}$:*

$$W(x, R) = \min_{x' \in U(x,R)} W(x', R^b).$$

The previous theorem offers a nice dual result to the characterization of the well-being measure satisfying *Lower Union Inclusion*. It is interesting and maybe surprising that *Lower Union Inclusion* and *Convex Hull Inclusion* have similar implications. Comparability among preferences is obtained by relationship to some reference preferences. In the former case, the reference preferences always give a lower bound on well-being. In the latter case, they give an upper bound.

Again, *Best Preferences* forces us to choose one preference relation, but there is no restriction on that choice. It is natural, though, to consider that some preferences are better candidates than others. We think that linear preferences are natural candidates. Preferences R^p are linear if there exist K parameters $p_k \in \mathbb{R}$, $k \in K$, such that

$$x \, R^p \, x' \Leftrightarrow \sum_{k \in K} p_k x_k \geq \sum_{k \in K} p_k x'_k.$$

An agent with linear preferences has the highest ability to substitute one good for another. All goods are equally valuable, whatever the proportion in which they come, as soon as we weight them with the p_k parameters.

If we take linear preferences for the best preferences, then the well-being measures characterized in the above theorem are ordinally equivalent to the money-metric utility, introduced by Samuelson (1974) and Samuelson and Swamy (1974). In their definition, the p vector stands for a vector of prices, and the money-metric utility at (x, R) is the minimal expenditure a consumer with preferences R would incur, facing price vector p, to reach the same satisfaction as at x. Instead on relying on the expenditure function terminology, we can define that well-being measure W^p by using the following function: for a set of bundles $B \in X$, for $R \in \mathcal{R}$, we write $\max(R, B)$ to denote any bundle in B that maximizes R over B, that is, $\max(R, B) = x$ only if $x \in B$ and $x \, R \, x'$ for all $x' \in B$. For all $x \in X$, all $R \in \mathcal{R}$,

$$W^p(x, R) = w \Leftrightarrow x \, I \, \max(R, \{x' \in X | px' \leq w\}).$$

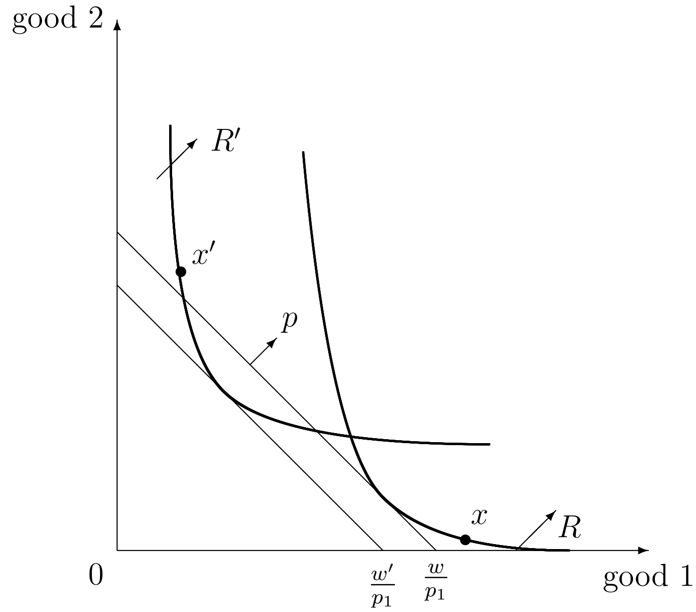

Figure 6: Illustration of W^p: $W^p(x, R) = w, W^p(x', R') = w'$.

Our result teaches us the following lessons about money-metric utility. First, it gives it some axiomatic foundation. Second, it offers us another interpretation of the p parameters. They are not only the reference price vector at which expenditures have to be valued. They are also the parameters of the linear preferences that the observer chooses as best preferences.

The literature on money-metric utility has failed to provide a convincing way of choosing the reference price vector. Our study does not allow us to make any progress on that issue. One would need more information on the different goods to be able to choose among the possible p's.

This theorem provides us with axiomatic justification to using money-metric utility as a well-being measure, but it justifies more than that: Money-metric well-being measures are only some among the many other measures that are covered by the theorem. We believe that linear preferences are natural candidates to play the role of best preferences, but we have no axiomatic argument to prefer them over other preferences.

The literature on fair allocation has often justified the equal income Walrasian allocation rule as a prominently fair one. The relationship with our result is clear. That rule amounts to equalize the well-being of all agents when one uses the well-being measure characterized by either *Convex Hull Inclusion* or *Nested Contour* and *Best Preferences* and the best preferences are linear, with p equal to the equilibrium prices. That may be viewed as a possible way of choosing a precise price vector to evaluate agents' well-being at allocations that are not competitive equilibrium allocations (see, in particular, Fleurbaey and Maniquet 2008). Unfortunately, it is not that clear. The literature on social ordering functions, indeed, has shown that other, non equilibrium, prices could be preferred to evaluate allocations that are not competitive equilibrium allocations. Again, we need to keep in mind that these two bodies of literature try to aggregate well-being at the same time as they try to construct well-being comparisons. Again, the important conclusion is that by disentangling the two issues and focussing on the construction of interpersonal comparisons, we have to keep a larger set of potential well-being measures.

5 Strict Well-Being Monotonicity

We now move to our third and last strengthening of *Nested Contour*. The axiom we define in this section is consistent with the idea that the well-being of an agent consuming some bundle of goods should be determined by the ranking of that good in the preference of the agent. It is hard to define rankings in a case of many perfectly divisible goods. We therefore use the following idea. If a bundle is obviously higher in the ranking of an agent than in the ranking of another agent, in the sense that the lower contour set of the former agent contains that of the latter, then the well-being of the former agent should be strictly larger.

Axiom 8 STRICT WELL-BEING MONOTONICITY
For all $x \in X,\ R, R' \in \mathcal{R}$*, if* $L(x, R') \subset L(x, R)$ *then* $W(x, R) > W(x, R')$*.*

The strict inequality is absolutely crucial. If only a weak inequality is required, then all the well-being measures we have been studying up to now would satisfy the axiom, as it would be a consequence of *Nested Contour*. If one takes a well-being measure satisfying *Worst Preferences*, for instance, with Leontieff preferences R^l as worst preferences, we may have $L(x, R) \subset L(x, R')$ and $W(x, R) = W(x, R')$ if $x = \lambda l$ for some $\lambda \in \mathbb{R}$. If one take a well-being measure satisfying *Best Preferences*, for instance, with

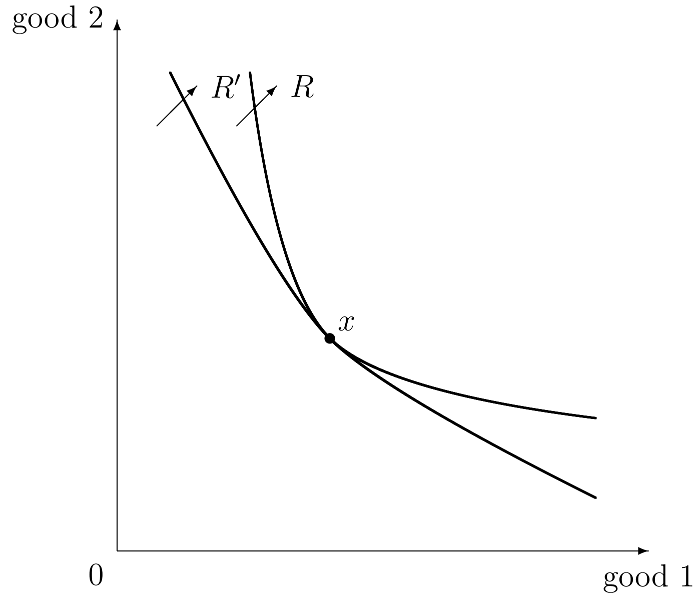

Figure 7: *Strict Well-Being Monotonicity*: $W(x, R) > W(x, R')$.

linear preferences R^p as worst preferences, we may have $L(x, R) \subset L(x, R')$ and $W(x, R) = W(x, R')$ if x is precisely the best bundle of both R and R' over the budget determined by income px.

Strict Well-Being Monotonicity is logically stronger than *Nested Contour*.

Lemma 3 *If a well-being measure satisfies* Strict Well-Being Monotonicity*, then it satisfies* Nested Contour*.*

Here is an example of a well-being measure satisfying *Strict Well-Being Monotonicity*. If we define the well-being of an agent as the K-dimensional volume of the lower contour set of that agent at the bundle she consumes, then the axiom is satisfied. Such a volume is always finite when X is compact. There are, of course, many other well-being measures that satisfy *Strict Well-Being Monotonicity*. It is difficult, though, to identify their properties precisely. The drawback of such measures can be that they are too sensitive to changes in preferences that take place in subsets of the consumption sets that are intuitively less likely to be the actual consumption of agents (think of bundles at which all coordinates but one are zero). For this reason, we do not investigate these measures any further.

6 Changing the nature of the goods

The conclusions we reached in the previous sections were driven by our assumptions on the nature of the goods. By changing our assumptions, we will show in this section that the formal results obtained so far do no longer hold, and new axioms are necessary.

We drop the assumption that all goods are cardinal and that more of them is always better. Health, for instance, plays a crucial role in the well-being of agents. Health, however, is not cardinally measurable. It does not make sense, indeed, to take the average between two different health levels. It is no longer clear, moreover, that better health is always unanimously preferred. Not all agents look for the most healthy diet or the most healthy sport practice. A lot of people content themselves with a good, not perfect, health.

In addition to the labor time and the wage, the quality of one's job also affects one's well-being. There is no consensus, though, about what is a better job. Tastes regarding jobs are different.

The location of one's housing is also a matter of taste. For the same set of prices, some would prefer to live close to a city center, whereas others would prefer to live far away.

The model we study in this section tries to capture the different natures of the good that impact agents' well-being. In order to do it with a simple model, we assume that there are two goods. The first good is divisible, cardinal, and more of it is always preferred to less. It is a representative good of all the goods of the previous model. The second good may be indivisible, ordinal and exhibit satiation. We assume there is a set A of attributes, which is a compact subset of the real line. A bundle is now a pair $(x, a) \in \mathbb{R}_+ \times A$. The set of admissible preferences $\mathcal{R}$ is the set of preferences that are the projection on X of continuous preferences on $\mathbb{R}^2_+$. We further assume that 1) preferences are monotonic in the first good: for all $(x, a) \in \mathbb{R}_+ \times A$, all $R \in \mathcal{R}$, $x > x'$ implies that $(x, a)\,P\,(x', a)$, and 2) no attribute a can be infinitely better than another attribute, that is, for all $(x, a) \in \mathbb{R}_+ \times A$, all $a' \in A$, all $R \in \mathcal{R}$, there exists $x' \in \mathbb{R}_+$ such that $(x', a')\,P\,(x, a)$.

The first difference between this model and the previous one is that *Lower Contour Inclusion* does no longer imply *Worst Preferences*. There are now well-being measures that satisfy the former axiom but not the latter. Here is an example. Let $\tilde{a} \in A$ be a fixed reference parameter. The well-being measure defined by $W((x, a), R) = w$ if and only if $(x, a)\,I\,(w, \tilde{a})$ satisfies

Lower Contour Inclusion and even *Intermediary Preferences*. There is no preferences, however, that have the property of being worst than all the others in the sense defined in the previous sections.

The Leontieff preferences, which were particularly relevant in the previous sections, are hardly defined in this model, as the minimum operator is no longer meaningful over A. One may wonder, therefore, what would be the natural worst preferences in the new model. A natural candidate consists in saying that consuming a bundle (x, a) is the most painful experience for the agent who considers a the worst possible attribute. That a is the worst possible attribute could be captured by assuming that any other attribute, combined with however low an element of $\mathbb{R}_+ \times A$ is better than consuming a. Unfortunately, no continuous preferences can represent such preferences. On the other hand, if we would have allowed for non-continuous preferences and we would have taken the lexicographic preferences according to which first not consuming a is valued and then consuming more of the other good is preferred, the associated axiom of *Worst Preferences* would precisely give rise to the well-being measure defined above, with $\tilde{a}$ being the worst attribute.

In conclusion, even if there is no axiomatic counterpart to Theorem 1 for this model, we can claim that selecting a reference attribute $\tilde{a}$ and measuring well-being by reference to the bundle containing $\tilde{a}$ to which the agent is indifferent is the natural generalization of that result. All well-being measures constructed this way are ordinally equivalent to the following $W^{\tilde{a}}$ measure: for all $(x, a) \in \mathbb{R}_+ \times A$, all $R \in \mathcal{R}$,

$$W^{\tilde{a}}(x, a) = w \Leftrightarrow (x, a)\, I\, (w, \tilde{a}).$$

The second difference between this model and the previous one is that *Convex Hull Inclusion* is no longer well defined. This comes from the fact that, without cardinality of the second good, the convex hull operation is no longer defined. The axiom of *Best Preferences*, on the other hand, continues to be relevant, and the second part of Theorem 3 continues to hold. The question of which preferences are a natural candidate to stand as best preferences can therefore be addressed again, even if linear preferences are no longer well defined either. We propose an axiomatic answer to that question.

We now introduce our last axiom. It captures the idea that there is a situation in which it is natural to claim that two agents have the same well-being level. Let us assume two agents, characterized by preferences R and

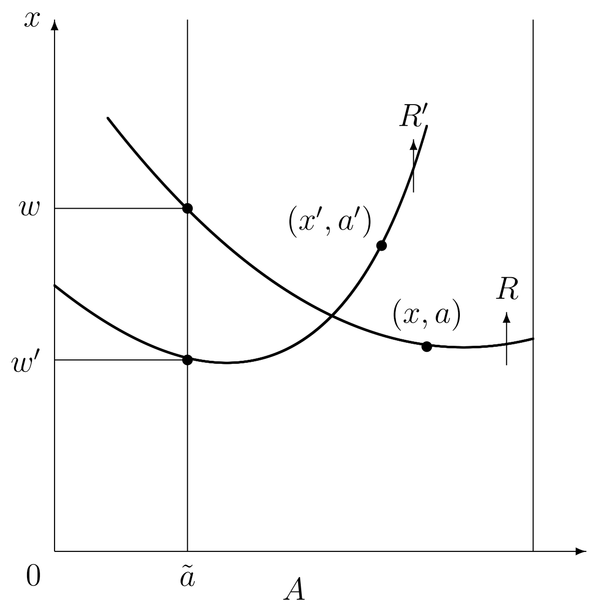

Figure 8: Illustration of $W^{\tilde{a}}$: $W^{\tilde{a}}((x,a),R) = w, W^{\tilde{a}}((x',a'),R') = w'$.

$R' \in \mathcal{R}$, consume the same bundle (x,a). Moreover, let us assume that a is the preferred attribute of both of them, in the sense that, at a level of consumption x of the first good, the satisfaction of either agent is maximized at a. Then both agents should be declared equally well-off. Assume it is not the case and we apply an egalitarian aggregator to these two agents. The conclusion would be that we need to redistribute the first good among them, to compensate the worst-off agent. Compensate for what, if she has her preferred attribute? That would be hard to justify.

It is convenient to use the following terminology in the definition of the axiom. For $x \in \mathbb{R}_+$ and $R \in \mathcal{R}$, we write $a_{\max}(x,R)$ to denote the set of preferred attributes of agent R when she consumes the quantity x of the first good, that is, $(x, a_{\max}(x,R))\, R'\, (x,a')$ for all $a' \in A$.

Axiom 9 EQUAL WELL-BEING AT PREFERRED ATTRIBUTE
For all $(x,a) \in \mathbb{R}_+ \times A$, all $R, R' \in \mathcal{R}$, if $a \in a_{\max}(x,R)$ and $a \in a_{\max}(x,R')$, then $W((x,a),R) = W((x,a),R')$.

Our last result characterizes the family of well-being measures that satisfy *Equal Well-Being at Preferred Attribute.* It echoes a similar characterization

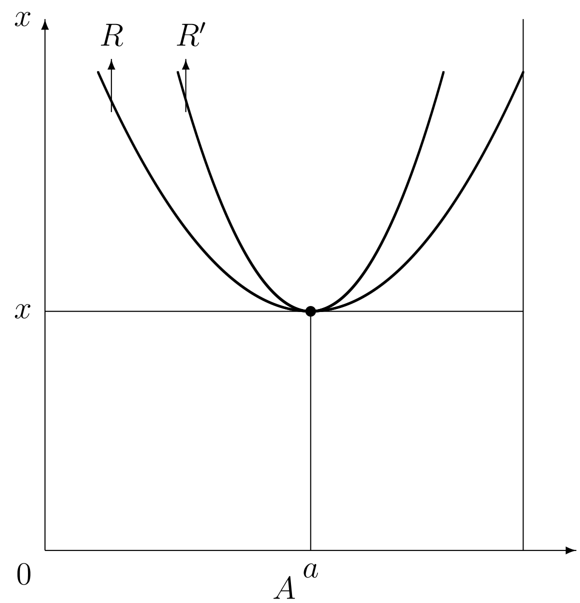

Figure 9: *Equal Well-Being at Preferred Attribute*: $W((x,a),R) = W((x,a),R')$.

developed by Fleurbaey and Blanchet (2013).

Theorem 4 *A well-being measure* W *satisfies* Equal Well-Being at Preferred Attribute *if and only if for all* $(x, a), (x', a') \in \mathbb{R}_+ \times A$*, all* $R, R' \in \mathcal{R}$*,*

$$\min_{(y,b)\in U((x,a),R)} y = \min_{(y',b')\in U((x',a'),R')} y' \Leftrightarrow W((x, a), R) = W(x', a'), R').$$

The well-being measures defined in the theorem satisfy *Best Preferences*, if the best preferences are defined in the following way: for all $(x, a), (x', a') \in \mathbb{R}_+ \times A$, $(x, a)\, R^b\, (x', a')$ if and only if $x \geq x'$, that is, the reference agent is indifferent among all possible attributes. We can clearly argue that it is quite a natural candidate for best preferences. This agent never needs to be compensated for not having an acceptable attribute because she considers that all attributes are equally good. The characterization of Fleurbaey and Blanchet (2013) is based on the axiom that these preferences are the best ones. All the well-being measures satisfying the axiom of the theorem are ordinally equivalent to $W^{a_{\max}}$ defined by: for all $(x, a) \in \mathbb{R}_+ \times A$, all $R \in \mathcal{R}$,

$$W^{a_{\max}}(x, a) = w \Leftrightarrow (x, a)\, I\, (w, a_{\max}(x, R)).$$

7 The general case

It may be tempting to interpret good x in the model of the previous section as income. For all the reasons listed above, that is, heterogeneity of prices facing agents, market imperfection or the existence of non-marketable goods, it is not legitimate to start at the level of the preferences of the agents for income. The question is now to combine the two well-being measures defined in the previous section with the ones defined above and which aim at providing a suitable alternative to income. This section shows that the combination is quite easy.

Let the consumption set of the agents now be $X \times A$, with $X \subseteq \mathbb{R}^K_+$ being the set of possible consumptions of divisible goods that are cardinally measurable and for which more is always better, and $A \subset \mathbb{R}$ being a compact set of attributes. Our objective is to combine the well-being measures W^ℓ and W^p defined over X and $W^{\tilde{a}}$ and $W^{a_{\max}}$ defined over $\mathbb{R}_+ \times A$. The main message of this section is that this gives us four different well-being measures,

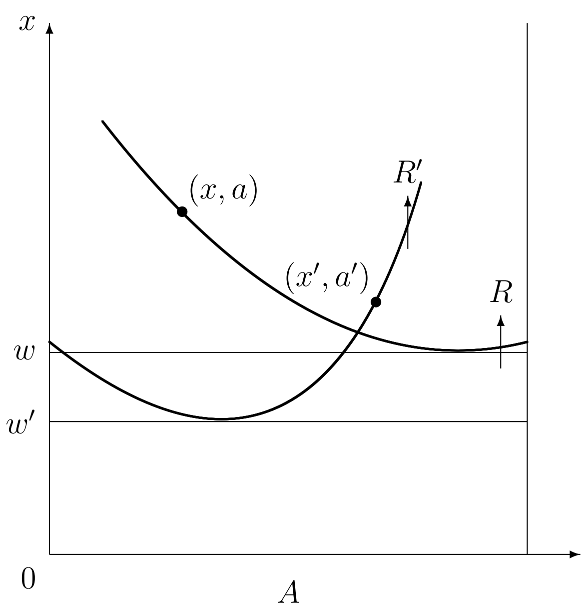

Figure 10: Illustration of $W^{a_{\max}}$: $W^{a_{\max}}((x,a),R) = w, W^{a_{\max}}((x',a'),R') = w'$.

as either of the former two measures can be combined with either of the latter two. Let us review the resulting measures in turn.

1. Combining W^{ℓ} and $W^{\tilde{a}}$, we can define $W^{\ell\tilde{a}}$ as follows: for all $(x,a) \in X \times A$, all $R \in \mathcal{R}$,

$$W^{\ell\tilde{a}}(x,a) = w \Leftrightarrow (x,a)\, I\, (w\ell, \tilde{a}).$$

2. Combining W^{ℓ} and $W^{a_{\max}}$, we can define $W^{\ell a_{\max}}$ as follows: for all $(x,a) \in X \times A$, all $R \in \mathcal{R}$,

$$W^{\ell a_{\max}}(x,a) = w \Leftrightarrow (x,a)\, I\, (w\ell, a_{\max}(w\ell, R)).$$

3. Combining W^{p} and $W^{\tilde{a}}$, we can define $W^{p\tilde{a}}$ as follows: for all $(x,a) \in X \times A$, all $R \in \mathcal{R}$,

$$W^{p\tilde{a}}(x,a) = w \Leftrightarrow (x,a)\, I\, \max\left(R, \{(x', \tilde{a}) \in X \times A | px' \leq w\}\right).$$

4. Combining W^{p} and $W^{a_{\max}}$, we can define $W^{pa_{\max}}$ as follows: for all $(x,a) \in X \times A$, all $R \in \mathcal{R}$,

$$W^{pa_{\max}}(x,a) = w \Leftrightarrow (x,a)\, I\, \max\left(R, \{(x', a') \in X \times A | px' \leq w, a' \in A\}\right).$$

8 Proofs

Proof of Lemma 1: 1) *Nested Contour* $\Rightarrow$ *Unchanged Indifference Surface Independence*. Let W satisfy *Nested Contour*. Let $x \in X$ and $R, R' \in \mathcal{R}$ be such that $I(x,R) = I(x,R')$. Assume $W(x,R) \neq W(x,R')$. Let us consider, without loss of generality, that $W(x,R) > W(x,R')$. By continuity of W, there exists $x' \in X$ such that $x\, P\, x'$ and yet $W(x',R) > W(x,R')$. Note that $I(x,R) = I(x,R')$ implies that $U(x,R') \cap L(x',R) = \emptyset$. By *Nested Contour*, $W(x',R) < W(x,R')$, a contradiction.

2) *Unchanged Indifference Surface Independence* $\Rightarrow$ *Nested Contour*. Let $x, x' \in X$ and $R, R' \in \mathcal{R}$ be such that $U(x,R) \cap L(x',R') = \emptyset$. Let $R'' \in \mathcal{R}$ be such that $I(x,R) = I(x,R'')$ and $I(x',R') = I(x',R'')$. By *Unchanged Indifference Surface Independence*, $W(x,R) = W(x,R'')$ and $W(x',R') = W(x',R'')$. By the assumption that W respects preferences, $W(x,R'') > W(x',R'')$. By transitivity, $W(x,R) > W(x',R')$, which proves the claim. ■

Proof of Theorem 1: We begin by proving the following lemma.

Lemma 4 *If a well-being measure W satisfies* Lower Contour Inclusion, *then for all $n \in \mathbb{N}$, $x^1 \dots, x^n, x'' \in X$, $R^1, \dots, R^n, R'' \in \mathcal{R}$, if*

$$(L(x'', R'') \cap X) \subset interior[(\cup_{i \in \{1,\dots,n\}} L(x^i, R^i)) \cap X],$$

then

$$W(x'', R'') < \max_{i \in \{1,\dots,n\}} \{W(x^i, R^i)\}.$$

Proof of Lemma 4: *Lower Contour Inclusion* is equivalent to the above property in the case $n = 2$. Assume the property holds up to $n - 1$. We claim it must hold for n as well. Assume

$$L(x'', R'') \subset interior[\cup_{i \in \{1,\dots,n\}} L(x^i, R^i)].$$

Let $x^{12} \in X$, $R^{12} \in \mathcal{R}$ be such that

$$(L(x^{12}, R^{12}) \cap X) \subset interior[(L(x^1, R^1) \cup L(x^2, R^2)) \cap X]$$

and yet

$$(L(x'', R'') \cap X) \subset interior[(\cup_{i \in \{12,3\dots,n\}} L(x^i, R^i)) \cap X].$$

By *Lower Contour Inclusion*,

$$W(x^{12}, R^{12}) < \max\{W(x^1, R^1), W(x^2, R^2)\}.$$

By the above property, holding up to $n - 1$,

$$W(x'', R'') < \max_{i \in \{12,3\dots,n\}} \{W(x^i, R^i)\}.$$

Gathering the last two inequalities, we get

$$W(x'', R'') < \max_{i \in \{1,\dots,n\}} \{W(x^i, R^i)\},$$

the desired outcome. **End of the proof of the lemma**

1) *Lower Contour Inclusion* $\Rightarrow$ *Nested Contour*: It follows from the definition of *Lower Contour Inclusion* applied to $x' = x$ and $R' = R$.

2) *Lower Contour Inclusion* $\Rightarrow$ *Worst Preferences*: The proof consists of constructing the worst preferences R^w. Let $r \in \mathbb{R}$. For $R \in \mathcal{R}$, let $r(R)$ be any $x \in \mathbb{R}_+^K$ such that $W(x, R) = r$. Let $U^r \subseteq \mathbb{R}_+^K$ be defined as

$$U^r = \bigcap_{R \in \mathcal{R}} U(r(R), R).$$

Let $x^r \in \mathbb{R}^K_+$, $R^r \in \mathcal{R}$ be such that $U(x^r, R^r) = U^r$. Such R^r exists because U^r is closed and convex and $\mathcal{R}$ contains all continuous, convex and monotone preferences over $\mathbb{R}^K_+$. We claim that $W(x^r, R^r) = r$. Assume not. If $W(x^r, R^r) < r$, then there exists $x \in \mathbb{R}^K_+$ such that $x \, P^r \, x^r$, $W(x, R^r) = r$ and $U(x, R^r) \subset interior[U^r]$, in contradiction to the way U^r is constructed. If $W(x^r, R^r) > r$, then, by continuity of W, there exists $x \in \mathbb{R}^K_+$ such that $x^r \, P^r \, x$, $W(x, R^r) > r$ and

$$L(x, R^r) \subset interior \left[\bigcup_{R \in \mathcal{R}} L(r(R), R) \right].$$

This implies

$$(L(x, R^r) \cap X) \subset interior \left[\left(\bigcup_{R \in \mathcal{R}} L(r(R), R) \right) \cap X \right].$$

Given that X is compact, there exists a finite set of preferences $R^1, \ldots, R^n \in \mathcal{R}$ such that

$$(L(x, R^r) \cap X) \subset interior \left[\left(\bigcup_{i \in \{1, \ldots, n\}} L(r(R^i), R^i) \right) \cap X \right].$$

By the lemma above, that implies that

$$W(x, R^r) < \max_{i \in \{1, \ldots, n\}} W(r(R^i), R^i) = r,$$

the desired contradiction.

By repeating the argument for all $r \in \mathbb{R}$, we obtain a set of nested convex upper contour sets, which, by continuity of the preferences, form a continuous, convex and monotone preference relation, which, given the universal domain assumption, belongs to $\mathcal{R}$.

3) Assume W satisfies *Nested Contour* and *Worst Preferences*. Let R^w denote the worst preferences, $x \in X$ and $R \in \mathcal{R}$. We claim that

$$W(x, R) = \max_{x' \in L(x,R)} W(x', R^w).$$

Assume not. Let $W(x, R) < \max_{x' \in L(x,R)} W(x', R^w) = m$. Let $x^* \in L(x, R)$ be such that $W(x^*, R^w) = m$. We have $W(x^*, R) < m = W(x^*, R^w)$, a contradiction to *Worst Preferences*. Let $W(x, R) > \max_{x' \in L(x,R)} W(x', R^w) =$

m. By continuity of W, there exists $x^* \in X$ such that $x \, P \, x^*$, $W(x^*, R) < W(x, R)$, and yet $W(x^*, R) > \max_{x' \in L(x,R)} W(x', R^w) = m$, in contradiction to *Nested Contour*.

4) *Worst Preferences* and *Nested Contour* $\Rightarrow$ *Lower Contour Inclusion*: Let $x, x', x'' \in X$ and $R, R', R'' \in \mathcal{R}$ be such that

$$(L(x'', R'') \cap X) \subset interior[(L(x, R) \cup L(x', R')) \cap X]. \tag{1}$$

We need to show that $W(x'', R'') < \max\{W(x, R), W(x', R')\}$. Let $\hat{x}, \hat{x}', \hat{x}'' \in X$ be such that

$$\begin{aligned} W(\hat{x}, R^w) &= \max_{\bar{x} \in L(x,R)} W(\bar{x}, R^w), \\ W(\hat{x}', R^w) &= \max_{\bar{x} \in L(x',R')} W(\bar{x}, R^w), \\ W(\hat{x}'', R^w) &= \max_{\bar{x} \in L(x'',R'')} W(\bar{x}, R^w). \end{aligned}$$

Note that

$$\max_{\bar{x} \in (L(x,R) \cup L(x',R'))} W(\bar{x}, R^w) = \max\{W(\hat{x}, R^w), W(\hat{x}', R^w)\}.$$

By Eq. 1,

$$W(\hat{x}'', R^w) < \max_{\bar{x} \in (L(x,R) \cup L(x',R'))} W(\bar{x}, R^w).$$

By what is proven in step 3) above, we get the desired outcome. ■

Proof of Lemma 2: Let $x, x', x'' \in X$ and $R, R', R'' \in \mathcal{R}$ be such that $(L(x'', R'') \cap X) \subset interior[(L(x, R) \cup L(x', R')) \cap X]$. We need to distinguish between two cases. Case 1: $(L(x, R) \cap X) \subset interior[L(x', R') \cap X]$. Then, $(L(x'', R'') \cap X) \subset interior[L(x', R') \cap X]$ as well. By *Nested Contour*, $W(x'', R'') < W(x', R')$, which proves the claim. The case $(L(x', R') \cap X) \subset interior[L(x, R) \cap X]$ is similar.

Case 2: neither $(L(x, R) \cap X) \subset interior[L(x', R') \cap X]$ nor $(L(x', R') \cap X) \subset interior[L(x, R) \cap X]$. Then, there exists $x^* \in X$ such that $x^* \in I(x, R) \cap I(x', R')$. We can find $R^* \in intermediary(R, R', x)$ such that $(L(x'', R'') \cap X) \subset interior[L(x^*, R^*) \cap X]$. By *Intermediary Preferences* $W(x^*, R^*) \leq \max\{W(x, R), W(x', R')\}$. By *Nested Contour*, $W(x'', R'') < W(x^*, R^*)$. Gathering the last two inequalities,

$$W(x'', R'') < \max\{W(x, R), W(x', R')\},$$

the desired outcome. ■

Proof of Theorem 2: By Lemma 2, W satisfies *Lower Contour Inclusion*. By Theorem 1, W satisfies *Worst Preferences*. Let R^w denote the worst preferences. The there boils down to claiming that all upper contour sets of R^w are of the shape: there exists $x \in X$ such that $U(x, R^w) = \{x' \in X | x' > x\}$. Assume R^w does not have this property. Then, there exists $x, x' \in X$ and $R, R' \in \mathcal{R}$ such that $W(x, R) = \max_{\bar{x} \in L(x,R)} W(\bar{x}, R^w)$, $W(x', R') = \max_{\bar{x} \in L(x',R')} W(\bar{x}, R^w)$, $W(x, R^w) = W(x', R^w)$ and yet $W(x, R) \neq W(x', R)$ and $W(x, R') \neq W(x', R')$. This is illustrated in the following figure. Let $x'' \in I(x, R) \cap I(x', R')$. Let $R'' \in intermediary(R, R', x'')$. By Theorem 1, $W(x, R) = W(x, R^w) = W(x', R^w) = W(x', R')$. By *Intermediary Preferences*, $W(x'', R'') = W(x'', R) = W(x'', R')$. Gathering the previous equalities, $W(x'', R'') = W(x, R^w)$, in contradiction to *Nested Contour*. ■

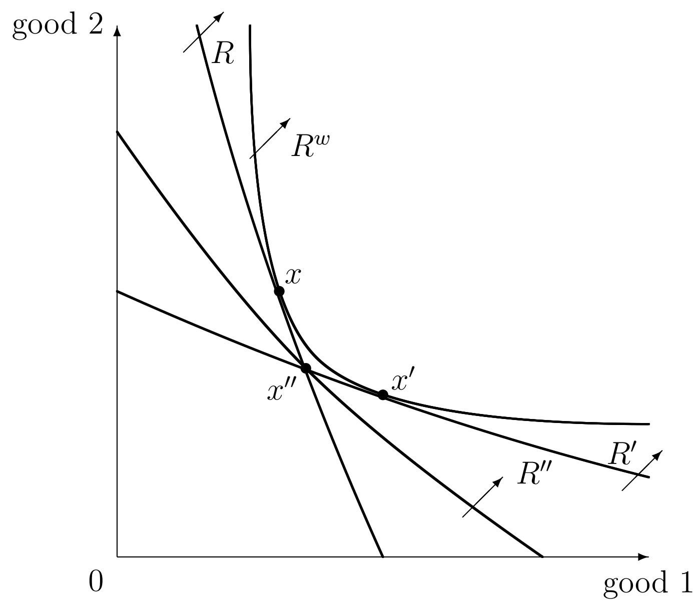

Figure 11: *Proof of Theorem 2*: illustration.

Proof of Theorem 3: This proof parallels the proof of Theorem 1. ■

Proof of Lemma 3: Let $x, x' \in X$ and $R, R' \in \mathcal{R}$ be such that $U(x, R) \cap L(x', R') = \emptyset$. Then, there exist $n \in \mathbb{N}$, $x^1, \ldots, x^{n+1} \in X$, $R^1, \ldots, R^n \mathcal{R}$ such that, $x' \, I' \, x^1$, $x^i \, I^i \, x^{i+1}$ for all $i \in \{1, \ldots, n\}$,

$$\begin{aligned} L(x^1, R') &\subset L(x^1, R^1), \\ L(x^{i+1}, R^i) &\subset L(x^{i+1}, R^{i+1}), \forall \, i \in \{1, \ldots, n-1\}, \\ L(x^{n+1}, R^n) &\subset L(x^{n+1}, R). \end{aligned}$$

The proof is illustrated in the next figure, for the case $n = 1$. By *Strict Well-Being Monotonicity*, $W(x^1, R') < W(x^1, R^1)$, $W(x^{i+1}, R^i) < W(x^{i+1}, R^{i+1})$ for all $i \in \{1, \ldots, n-1\}$, and $W(x^{n+1}, R^n) < W(x^{n+1}, R)$. As W respect preferences, $W(x', R') = W(x^1, R')$ and $W(x, R) = W(x^{n+1}, R)$. Gathering these inequalities and equalities, we obtain $W(x, R) > W(x', R')$, the desired outcome. ■

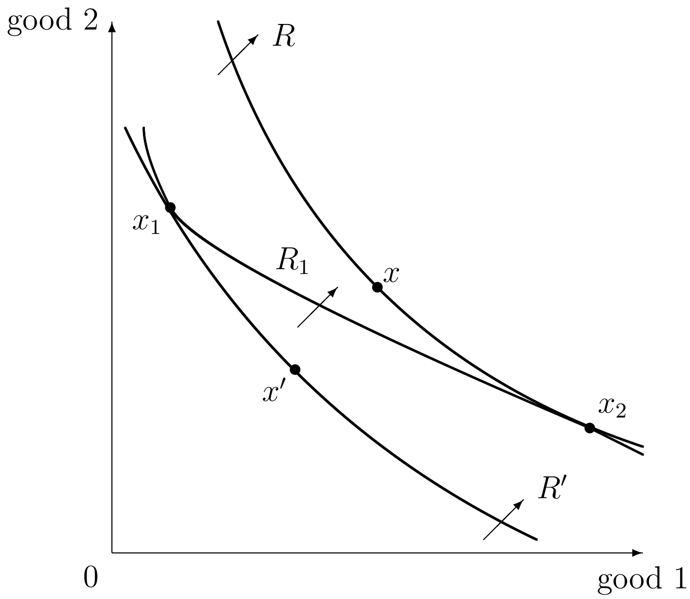

Figure 12: *Proof of Lemma 3*: illustration.

Proof of Theorem 4: Let $(x, a), (x', a') \in \mathbb{R}_+ \times A$ and $R, R' \in \mathcal{R}$ be such that $\min_{(y,b) \in U((x,a),R)} y \geq \min_{(y',b') \in U((x',a'),R')} y'$. Let $(\bar{x}, \bar{a}), (\bar{x}', \bar{a}') \in \mathbb{R}_+ \times A$ be such that $\bar{x} = \min_{(y,b) \in U((x,a),R)} y$, $\bar{x}' = \min_{(y',b') \in U((x',a'),R')} y'$, $(x, a) \, I \, (\bar{x}, \bar{a})$ and $(x', a') \, I' \, (\bar{x}', \bar{a}')$. As W respects preferences, $W((x, a), R) = W((\bar{x}, \bar{a}), R)$ and $W((x', a'), R') = W((\bar{x}', \bar{a}'), R')$. Let $\bar{R} \in \mathcal{R}$ be defined by: for all

$(y,b),(y',b') \in \mathbb{R}_+ \times A$,

$$(y,b)\,\bar{R}\,(y',b') \Leftrightarrow y \geq y'.$$

By *Equal Well-Being at Preferred Attribute*, $W((\bar{x},\bar{a}),\bar{R}) = W((\bar{x},\bar{a}),R)$ and $W((\bar{x}',\bar{a}'),\bar{R}) = W((\bar{x}',\bar{a}'),R')$. As W respects preferences, $W((\bar{x},\bar{a}),\bar{R}) \geq W((\bar{x}',\bar{a}'),\bar{R})$. Gathering these last inequalities and equalities, we obtain $W((x,a),R) \geq W(x',a'),R')$, the desired outcome. ■

9 Conclusion

Evaluating social policies or assessing the level of social welfare in an economy requires to define ways of aggregating individual measures of well-being. Among the many theories that propose ways of doing it, the theory of fair allocation has provided solutions based on the idea that economic justice is a matter of fair allocation of resources. These solutions are strongly egalitarian. In the study of social ordering functions, for instance, only maximin types of aggregators turn out to receive axiomatic justification.

Our line of research, in this paper, has been to disentangle the question of measuring individual well-being from the question of aggregating it at the social level. We have then reviewed some typical requirements that are used in the theory of fair allocation to impose them on a new, simple object, which we called well-being measure, which only aims at constructing interpersonal comparisons. Our first finding is that the well-being measures should take the nature of the goods into account. The key distinction is whether a good is divisible, cardinally measurable and desirable in the sense that more of it is always preferred, or not.

Our second finding is that in the former case our axioms turn out to justify two families of well-being measures, each of which contain measures that are common in the whole literature on well-being measurement and fair allocation. The axiomatic justification we give to these two families are dual to each other. One family is consistent with the idea that the fundamental choice that has to be made is about worst preferences: which preferences make experiencing this or that bundle the most painful one. It turns out that making such a choice allows one to fully compare well-being between any pair of agents. If worst preferences are of the Leontieff type, that is, if they are the preferences of someone who is unable to trade-off among goods, then the resulting measure is the ray utility measure of Samuelson (1977), Deaton

(1979), Pazner (1979), Pazner and Schmeidler (1978), Decancq, Fleurbaey and Maniquet (2013) and many others.

The other family is consistent with the idea that the fundamental choice is about best preferences: which preferences make experiencing this or that bundle the least painful one. Again, making such a choice allows one to fully compare well-being between any pair of agents. If best preferences are linear, then the resulting measure is the money-metric utility well-being measure of Samuelson (1974), Samuelson and Swamy (1974) and many others.

Our third finding is that in the latter case, a choice has to be made between two families of measures. In the first family, we need to choose a reference parameter and measure well-being with respect to the equivalent satisfaction level of agents should they consume that reference attribute. In the second family, we need to let agents choose their preferred attribute and measure well-being with respect to the equivalent satisfaction level should they consume their preferred attribute. Our fourth finding is that, in the general case in which some goods are of one nature and other goods are of the other nature, the two pairs of measures we justify can be mixed, which gives us four possible families of measures.

This paper presents a summary of the main conclusions about how to measure well-being when the assumption is that economic justice is a matter of fair resource allocation and agents have different preferences. Even if heterogenous preferences is one of the most challenging questions in the well-being measurement problem and certainly the one that has received the largest attention, it is not the only one. The question of how to measure well-being when agents have heterogenous needs and abilities is certainly worth being addressed as well.

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